1. An individual tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that

- when the defendant is guilty, each judge will independently vote guilty with probability 0.8.
- when the defendant is not guilty, each judge will independently vote guilty with probability 0.2.

Suppose that 70 percent of all defendants are guilty (thus, 30 percent of all defendants are not guilty).
(a) Find the probability that a defendant who is not guilty will be declared guilty by the panel.
(b) Find the probability that judge number 3 votes guilty given that judges 1 and 2 vote guilty.

2. Suppose $X \sim \text{nib}(r, p)$, where $r \geq 1$ is known and p is unknown. Recall that the probability mass function (pmf) of X is

$$f_X(x|p) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r}, & x = r, r+1, r+2, ..., \\ 0, & \text{otherwise}, \end{cases}$$

where $0 < p < 1$, and that the moment generating function (mgf) of X is

$$M_X(t) = \left(\frac{pe^t}{1 - qe^t} \right)^r,$$

for $t < -\ln q$, where $q = 1 - p$. In this model, X records the number of Bernoulli trials until the rth success is observed.
(a) Show that X has pmf in the exponential family.
(b) Find the pmf of $Y = X - r$.
(c) Define $Z_p = 2pY$, and let $M_{Z_p}(t)$ denote the mgf of Z_p. Show that

$$\lim_{p \to 0} M_{Z_p}(t) = \left(\frac{1}{1 - 2t} \right)^r,$$

for all $t < 1/2$. In the light of this result, describe what happens to the cumulative distribution function (cdf) of Z_p as $p \to 0$.

3. Suppose $X \sim \text{beta}(\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$. Recall that the probability density function (pdf) of X is

$$f_X(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \, x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1).$$

Throughout this problem, we will consider the subfamily of beta distributions where $\alpha = \beta$.
(a) Describe, geometrically, the parameter space for members of the subfamily; e.g., draw a

picture, etc.
(b) Suppose $k > 0$. Show that the kth **central** moment of X is

$$\mu_k = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \sum_{j=0}^{k} \binom{k}{j} \frac{\Gamma(\alpha+j)}{\Gamma(2\alpha+j)} \left(-\frac{1}{2}\right)^{k-j}.$$

(c) Show that $X \stackrel{d}{=} 1 - X$, where recall "$\stackrel{d}{=}$" means "equal in distribution."
(d) Find $\text{corr}(X, 1-X)$.

4. Suppose $X \sim \text{exponential}(\beta_1)$, $Y \sim \text{exponential}(\beta_2)$, and $X \perp\!\!\!\perp Y$. Let $Z = X + Y$.
(a) If $\beta_1 = \beta_2$, find the moment generating function (mgf) of Z and then identify the distribution of Z.
(b) If $\beta_1 \neq \beta_2$, then the mgf of Z will not be helpful in identifying the distribution of Z. Recall the following result dealing with convolving probability density functions (pdfs):

Result: If X and Y are independent continuous random variables with marginal pdfs $f_X(x)$ and $f_Y(y)$, respectively, then the pdf of $Z = X + Y$ is

$$f_Z(z) = \int_{\mathbb{R}} f_X(w) f_Y(z-w)\, dw.$$

Use this result to find the pdf of Z when $\beta_1 \neq \beta_2$.
(c) If $\beta_1 = \beta_2$, find a function of X and Y that has an F distribution. Identify the degrees of freedom associated with your function.

5. Suppose $X_1, X_2, ..., X_n$ are iid Pareto(α, β), where $\alpha > 0$ and $\beta > 0$. Recall that the Pareto probability density function (pdf) is given by

$$f_X(x) = \frac{\beta \alpha^\beta}{x^{\beta+1}}\, I(x > \alpha).$$

Let $X_{(1)}$ and $X_{(n)}$ denote the minimum and maximum order statistics, respectively.
(a) Show that $X_{(1)} \sim \text{Pareto}(\alpha, n\beta)$.
(b) Show that the cumulative distribution function (cdf) of $X_{(n)}$ is

$$F_{X_{(n)}}(x) = \left[1 - \left(\frac{\alpha}{x}\right)^\beta\right]^n, \quad x > \alpha.$$

(c) Define the sequence of real numbers $b_n = \alpha n^{1/\beta}$, for $n = 1, 2, ...,$. Determine

$$\lim_{n\to\infty} F_{X_{(n)}}(b_n x).$$

This limit turns out to be the cdf of the standard Fréchet distribution, a useful model in extreme value theory.

6. Suppose that the continuous random variable X, conditional on $P = p$, has a two-component normal mixture distribution; specifically, $X|P = p \sim f_X(x|p)$, where the probability density function (pdf)

$$f_X(x|p) = p f_0(x) + (1-p) f_1(x),$$

and where the component densities

$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2},$$

where $-\infty < \mu < \infty$. Both $f_0(x)$ and $f_1(x)$ are defined for all $x \in \mathbb{R}$. In turn, the random variable $P \sim \text{beta}(\alpha, 1)$, where $\alpha > 0$ is known.
(a) Find the marginal pdf of X.
(b) Using the marginal pdf of X in part (a) or otherwise, show that $E(X) = \mu/(\alpha + 1)$.
(c) Suppose that, after observing X, you need to make a decision about which distribution X comes from: $f_0(x)$ or $f_1(x)$. Formulate a rule that will enable you to do this, discuss why your rule makes sense, and then investigate the characteristics of your rule (e.g., how often you make the correct decision, etc.).

Remark: For specificity in part (c), you might pick a value of α; e.g., $\alpha = 10$. Also, make an assumption about whether $\mu > 0$ or $\mu < 0$. This will make things easier.