1. Suppose $(S, \mathcal{B}, P)$ is a probability space.
(a) Describe clearly what a sigma-algebra $\mathcal{B}$ is and what properties define a sigma-algebra.
(b) Using mathematical notation, list the three axioms that a probability set function P satisfies.
(c) Suppose that $A_1, A_2, ..., A_n$ is a finite sequence of events. Describe clearly what it means for $A_1, A_2, ..., A_n$ to be independent.
(d) Define mathematically what it means for X to be a random variable over $(S, \mathcal{B}, P)$.

2. A diagnostic test used to detect the presence of a disease D is known to give a false positive result 1% of the time; it correctly gives a positive result when the disease is present 98% of the time. The disease is known to be present in 1 out of every 1000 people in a large population.
(a) If a randomly selected person tests positively, what is the probability the person has the disease?
(b) If a randomly selected person tests positively on each of three test applications (given in sequence, say), what is the probability the person has the disease?

3. Suppose that $F_1(x)$ and $F_2(x)$ are cumulative distribution functions (cdfs). Suppose that X is a random variable with

$$F_X(x) = P_X(X \leq x) = p_1 F_1(x) + p_2 F_2(x),$$

for $x \in \mathbb{R}$, where $0 < p_1 < 1$, $0 < p_2 < 1$, and $p_1 + p_2 = 1$.
(a) Prove that $F_X(x)$ is a valid cdf. You may assume that $F_1(x)$ and $F_2(x)$ are valid.
(b) For this part, take

$$\begin{aligned} F_1(x) &= (1 - e^{-x^2}) I(x > 0) \\ F_2(x) &= (1 - e^{-x/5}) I(x > 0). \end{aligned}$$

Derive the probability density function of X.
(c) In part (b), find $E(X)$.

4. Suppose that X is a random variable with probability mass function (pmf)

$$f_X(x) = \begin{cases} \alpha p^x, & x = 1, 2, 3, ..., \\ 0, & \text{otherwise}, \end{cases}$$

where $0 < p < 1$ is a constant.
(a) What value of α makes $f_X(x)$ a valid pmf?

For parts (b) and (c), suppose that $f_X(x)$ is a probability model for X, the number of children a family will have.

(b) What is the mean number of children the family will have?
(c) If each child is equally likely to be a girl or boy (independently of each other; no multiple births, etc.), what is the probability that the family will have k girls (and any number of boys)?

5. Suppose X is a random variable with $E(X) = \mu$ and $\text{var}(X) = \sigma^2 < \infty$. Suppose the moment generating function (mgf) of X, $M_X(t)$, is defined for all $t \in (-h, h)$, where $h > 0$.

(a) What is $M_X(0)$?
(b) Suppose $a, b \in \mathbb{R}$. Derive the mgf of $Y = a + bX$ in terms of $M_X(t)$.
(c) Calculate

$$\lim_{t \to 0} \frac{\ln M_Y(t) - (a + b\mu)t}{t^2}.$$

6. Suppose that $X_1, X_2, ...,$ is a sequence of random variables. The probability density function (pdf) of X_n is

$$f_{X_n}(x) = nx^{n-1} I(0 < x < 1).$$

(a) Derive the cumulative distribution function (cdf) of X_n. Call it $F_{X_n}(x)$.
(b) Find the pointwise limit of the sequence of functions $F_{X_1}, F_{X_2}, ...$ Call the limit F_X. Is $F_X(x)$ a valid cdf? Explain.
(c) The moment-generating function (mgf) of X_n is given by

$$M_{X_n}(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{n+r}{n+r+1} \right) \frac{t^k}{k!}.$$

Suppose that $M_{X_n}(t) \to M_X(t)$ pointwise for all t, where $M_X(t)$ is the mgf of $X \sim F_X(x)$, defined in part (b). Find $M_X(t)$.