1. Suppose that X is a continuous random variable with probability density function (pdf)

$$f_X(x|\theta) = \frac{1}{\theta}e^{-x/\theta}\exp(-e^{-x/\theta}), \quad -\infty < x < \infty,$$

where $\theta > 0$ is an unknown parameter.
(a) Show that $\{f_X(x|\theta); \theta > 0\}$ is a scale family. Identify the scale parameter and the standard pdf.
(b) Derive the pdf of

$$Y = g(X) = e^{-X/\theta}$$

and identify the distribution of Y.

2. Suppose that X is a continuous random variable with probability density function (pdf)

$$f_X(x|\theta) = \frac{1}{\theta}\sqrt{\frac{2}{\pi}}e^{-x^2/2\theta^2} I(x > 0),$$

where $\theta > 0$ is an unknown parameter.
(a) Show that $\{f_X(x|\theta); \theta > 0\}$ is an exponential family. Is the family full or curved?
(b) Find $E(X)$ and $\mathrm{var}(X)$.

3. Suppose that (X, Y) is a continuous random vector with joint probability density function (pdf)

$$f_{X,Y}(x, y) = cx^2y \ I(0 < 2x < y < 1).$$

(a) Find c that makes $f_{X,Y}(x, y)$ a valid pdf. *Hint:* $c = 120$; now show it (draw a picture).
(b) Find both marginal pdfs.
(c) Find $E(X|Y)$.

4. Consider the hierarchical model

$$\begin{aligned} X|Y = y &\sim \text{geometric}(y) \\ Y &\sim \text{beta}(a, b), \end{aligned}$$

where $a > 0$ and $b > 0$. In the first model, conditional on $Y = y$, the random variable X counts the number of Bernoulli trials until the first success is observed. That is,

$$f_{X|Y}(x|y) = \begin{cases} (1-y)^{x-1}y, & x = 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the (marginal) probability mass function (pmf) of X.
(b) Calculate $E(X)$ and $\mathrm{var}(X)$. Note any additional restrictions on the values of a and b. **In this part, simplify as much as possible, but don't worry about tedious algebra.**

5. Suppose that $X \sim \mathcal{N}(0, 1)$, $Y \sim \mathcal{N}(0, 1)$, and $X \perp\!\!\!\perp Y$. Define

$$\begin{aligned} U &= X + Y \\ V &= X^2 + Y^2. \end{aligned}$$

(a) Show that the joint moment-generating function of U and V is given by

$$M_{U,V}(s,t) = E(e^{sU+tV}) = \frac{1}{1-2t} \exp\left(\frac{s^2}{1-2t}\right), \quad \text{for } t < 1/2.$$

(b) Find $\text{corr}(U, V)$.
(c) Are U and V independent? Explain.

6. Particles are subject to collisions that cause them to split into two parts with each part a fraction of the parent. Suppose that this fraction X is uniformly distributed over $(0, 1)$. Following a single particle through several splits, we obtain a fraction of the original particle

$$Z = \prod_{i=1}^{n} X_i.$$

Assume $X_1, X_2, ..., X_n$ are mutually independent and $X_i \sim \mathcal{U}(0, 1)$, for $i = 1, 2, ..., n$.
(a) Show that the probability density function (pdf) of Z is given by

$$f_Z(z) = \frac{1}{(n-1)!}(-\ln z)^{n-1} I(0 < z < 1).$$

(b) Calculate $\text{var}(Z)$.