GROUND RULES:

- This exam contains 6 questions. The questions are of equal weight.
- Print your name at the top of this page in the upper right hand corner.
- This exam is closed book and closed notes.
- Show all of your work and explain all of your reasoning! **Translation:** No work, no credit. Insufficient explanation, no credit. If you are unsure about whether or not you should explain a result or step in your derivation/proof, then this means you probably should explain it.
- Do not talk with anyone else about this exam. You must work by yourself. No communication of any type with others.

1. Suppose $(S, \mathcal{B}, P)$ is a probability space. Assume all sets below (e.g., A, B, C, $A_1, A_2, ...,$, etc.) are measurable with respect to $\mathcal{B}$.

(a) Describe what it means for a sequence of sets $A_1, A_2, ...,$ to "converge." What does this imply about $\lim_{n\to\infty} P(A_n)$?

(b) If $A \subseteq B$, prove that $P(A) \leq P(B)$. Then, for the set C with $P(C) > 0$, show that $P(A|C) \leq P(B|C)$.

(c) State the mathematical definition of what it means for $A_1, A_2, ..., A_n$ (a finite sequence) to be mutually independent.

(d) If $P(A_n) = 1$, for $n = 1, 2, ...,$ prove that $P(\cap_{n=1}^{\infty} A_n) = 1$ as well.

(e) Explain mathematically what it means for X to be a random variable over $(S, \mathcal{B})$. Also, describe how probabilities of events involving X can be calculated over $(S, \mathcal{B}, P)$.

2. Suppose X is a discrete random variable with probability mass function $f_X(x)$ given by

$$f_X(x) = pf_1(x) + (1-p)f_2(x),$$

where

$$\begin{aligned} f_1(x) &= I(x=0) \\ f_2(x) &= \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, ..., \end{aligned}$$

$0 < p < 1$, and $\lambda > 0$. In other words, X is a mixture of a degenerate distribution (at zero) and the usual Poisson distribution with mean λ.

(a) Show that

$$\begin{aligned} P(X=0) &= p + (1-p)e^{-\lambda} \\ P(X=x) &= (1-p)\frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 1, 2, ... \end{aligned}$$

(b) Derive $E(X)$ and $\text{var}(X)$.

(c) Derive $M_X(t)$, the moment generating function of X. What happens to $M_X(t)$ when $p \to 0$? Comment on what this implies.

(d) I used R to generate an iid sample of size $n = 30$ from $f_X(x)$:

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0 17  0  0 25  0  0 15  0  0  0  0 15  0  0  0  0  0  0  0  0 18  0 16 20  0 23  0  0  0
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What values of p and λ do you think I used to perform the simulation? Just guess and justify your answers. Don't do anything fancy here (you should be able to make a good educated guess).

3. Suppose X is a random variable with cumulative distribution function (cdf)

$$F_X(x|\lambda,\beta) = \begin{cases} 1 - e^{-(\lambda x)^\beta}, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where the parameters $\lambda > 0$ and $\beta > 0$.

(a) Find the probability density function $f_X(x|\lambda,\beta)$. Does X have pdf in the exponential family? What named distribution does X have when $\beta = 1$?

(b) Derive $E(X)$ and $\text{var}(X)$.

(c) Suppose $X_1, X_2, ..., X_n$ are iid from $F_X(x|\lambda,\beta)$. Derive $F_{X_{(1)}}(x) = P(X_{(1)} \leq x)$, the cumulative distribution function of the minimum order statistic $X_{(1)}$, and calculate $E(X_{(1)})$.

4. Consider the random vector (X, Y) with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{e^{-y}}{y}, & 0 < x < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write $P(Y < X^2)$ as a double integral. You do not have to calculate the integral, but I want to see a good picture explaining where the limits come from.

(b) Find $E(X)$ and $E(Y)$.

(c) Calculate the correlation of X and Y.

5. Suppose X_1, X_2, X_3 are mutually independent random variables where $X_1 \sim \chi^2_{r_1}$, $X_2 \sim \chi^2_{r_2}$, and $X_3 \sim \chi^2_{r_3}$.

(a) Under what condition are X_1, X_2, X_3 iid?

(b) Derive the joint probability density function (pdf) of

$$U_1 = \frac{X_1}{X_2} \quad \text{and} \quad U_2 = X_1 + X_2.$$

(c) Show that U_1 and U_2 are independent and that $U_2 \sim \chi^2_{r_1+r_2}$.

(d) Argue that

$$\frac{X_1/r_1}{X_2/r_2} \quad \text{and} \quad \frac{X_3/r_3}{(X_1+X_2)/(r_1+r_2)}$$

are independent and that each quantity has an F distribution. Your argument can consist of a well written description (without a lot of mathematics).

6. Suppose $X_1, X_2, ..., X_n$ is an iid sample from a $\mathcal{N}(0, \sigma^2)$ distribution, where $\sigma^2 > 0$. Define the statistics

$$T_1 = \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \quad \text{and} \quad T_2 = \frac{1}{n} \sum_{i=1}^{n} |X_i|.$$

(a) Derive the sampling distribution of T_1.

(b) Calculate $E(T_1)$ and $E(T_2)$ and establish an inequality between them.

(c) Calculate $\text{var}(T_2)$.