From Casella and Berger, do the following problems from Chapter 1:

Homework 1: 3, 6, 11, 12, 14, and 23.
Homework 2: 34, 35, 38, 39, 45, and 46.
Homework 3: 49, 51, 52, 53, 54, and 55.

These are extra problems that I have given on past exams (in STAT 712 or in related courses). You do not have to turn these in.

1.1. Define the sequence of sets $A_n = \{\omega : \frac{1}{n} < \omega < 1 + \frac{1}{n}\}$.
(a) Is $\{A_n\}$ monotone?
(b) Find

$$\bigcap_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n.$$

(c) Find $\limsup_n A_n$ and $\liminf_n A_n$. Does $\lim_{n\to\infty} A_n$ exist? Why or why not?
(d) Repeat parts (a-c) with $A_n = \{\omega : -n \leq \omega < 1 - \frac{1}{n}\}$.

1.2. Let $(S, \mathcal{B})$ be a measurable space and suppose that $C \in \mathcal{B}$. Show that the collection of sets defined by $\mathcal{B}_C = \{C \cap A : A \in \mathcal{B}\}$ is a σ-algebra on C. *Hint:* The complement of a set relative to $C \subseteq S$ is $C \cap A^c$, where A^c is the complement of A relative to S. Also if $C \in \mathcal{B}$, then $\mathcal{B}_C \subseteq \mathcal{B}$.

1.3. Suppose $(S, \mathcal{B})$ is a measurable space. Suppose P_1 and P_2 are probabilities on $(S, \mathcal{B})$. Suppose $0 \leq \alpha \leq 1$. For any $A \in \mathcal{B}$, define the set function

$$P(A) = \alpha P_1(A) + (1 - \alpha) P_2(A).$$

Show that P also defines a probability on $(S, \mathcal{B})$.

1.4. Suppose that $S = \{\omega \in \mathbb{R} : -1 < \omega < 2\}$ and let $\mathcal{B} = \{B \cap S : B \in \mathcal{B}(\mathbb{R})\}$ denote the collection of Borel sets on S. Define the sequence of events

$$A_n = \begin{cases} (-1/n, 0], & \text{if } n \text{ is odd;} \\ (0, 1 + 1/n], & \text{if } n \text{ is even.} \end{cases}$$

(a) True or False: $\{A_n, n = 1, 2, ...,\}$ partitions S.
(b) Does $\lim_{n\to\infty} A_n$ exist? If so, show that it does. If not, show why not.
(c) Suppose that P is a set function on $(S, \mathcal{B})$ defined by

$$P(A) = \int_A \frac{2}{9}(x+1)dx$$

for any $A \in \mathcal{B}$. Show that P satisfies the Kolmolgorov Axioms.

1.5. Suppose that $(S, \mathcal{B}, P)$ is a probability space and that $\{A_n\}$ is a sequence of measurable events; i.e., $A_n \in \mathcal{B}$, for $n = 1, 2, ...,$. Show that

$$P(A_n) = 1 \ \forall n \implies P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1.$$

Is the converse true? If so, prove it. If not, give a counterexample.

1.6. Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear.
(a) First show that

$$f_n = f_{n-1} + f_{n-2},$$

for $n \geq 2$, where $f_0 = 1$ and $f_1 = 2$.
(b) If P_n denotes the probability that successive heads never appear when a coin is tossed n times, find P_n (in terms of f_n) when all possible outcomes of the n tosses are assumed to be equally likely. Calculate P_{10}.
(c) Show that

$$\frac{f_n}{f_{n-1}} \to \frac{1+\sqrt{5}}{2}, \quad \text{as } n \to \infty.$$

(d) Show that, for large n,

$$P_n \approx \frac{P_{n-1}}{2}\left(\frac{1+\sqrt{5}}{2}\right).$$

1.7. Take $S = \{1, 2, ..., n\}$ and suppose that A and B, independently, are equally likely to be any of the 2^n subsets of S (including $\emptyset$ and S itself).
(a) Show that

$$P(A \subseteq B) = \left(\frac{3}{4}\right)^n.$$

Hint: Let $|B|$ denote the number of elements in B. Write $P(A \subseteq B)$ using LOTP where you condition on $\{|B| = i\}$, for $i = 0, 1, 2, ..., n$. *Note:* When I write $A \subseteq B$, I emphasize that A and B could be equal.
(b) Under identical conditions, what is $P(A \subset B)$? *Note:* When I write $A \subset B$, I emphasize that A and B can not be equal; i.e., A is a proper subset of B.

1.8. Suppose $(S, \mathcal{B}, P)$ is a probability space and let A and B be measurable events; i.e., $A \in \mathcal{B}$ and $B \in \mathcal{B}$.
(a) If $P(A \cap B) = 0$ and $P(A) = 1$, show that $P(B) = 0$.
(b) If $P(A) > 0$, $P(B) > 0$, and $P(A) < P(A|B)$, show that $P(B) < P(B|A)$.
(c) If A and B are independent, show that A^c and B^c are independent.

1.9. Suppose X is a continuous random variable with cdf F_X. Suppose $a \geq 0$. Prove that

$$\int_{-\infty}^{\infty} \{F_X(x+a) - F_X(x)\} dx = a.$$

1.10. A random variable X is said to be *symmetric* if X and $-X$ have the same probability distribution, that is, if $X \stackrel{d}{=} -X$.
(a) Show that X is symmetric if and only if $P_X(X \leq -x) = P_X(X \geq x)$, for all $x \in \mathbb{R}$.
(b) Obtain an equivalent condition for symmetry of X in terms of its cdf F_X.

(c) If X is symmetric, show that $P_X(|X| \leq x) = 2F_X(x) - 1$, for all $x \geq 0$.
(d) If X is symmetric, show that $P_X(X \geq 0) = 0.5$. Assume X is continuous.
(e) Suppose X has pdf f_X. Prove that X is symmetric if and only if f_X is an even function.

1.11. Suppose that F_1, F_2, ..., F_k are cdfs. Also let $p_1, p_2, ..., p_k$ be positive real numbers with $\sum_{i=1}^k p_i = 1$. Define $G_X(x) = \sum_{i=1}^k p_i F_i(x)$. G_X refers to a *mixture distribution.*
(a) Prove that G_X is a valid cdf.
(b) For specificity, take $k = 2$, $F_1(x) = (1 - e^{-x^2})I(x > 0)$, $F_2(x) = (1 - e^{-x/10})I(x > 0)$, and $p_1 = p_2 = 0.5$. That is, the distribution of X is a mixture of 2 distributions (with equal weighting). Derive the pdf of X and show that it integrates to 1. Graph the pdf. What happens to the pdf when the weights (p_1 and p_2) change?

1.12. Suppose that $F_0(x)$ is a cumulative distribution function (cdf). Suppose that X is a random variable with

$$F_X(x) = P_X(X \leq x) = (1 + \delta)F_0(x) - \delta[F_0(x)]^2,$$

for $x \in \mathbb{R}$, where δ satisfies $-1 \leq \delta \leq 1$.
(a) Show that $F_X(x)$ is a valid cdf. You may assume that $F_0(x)$, the so-called "base cdf," is valid.
(b) For this part, take the base cdf to be $F_0(x) = 1 - e^{-x/\lambda}$, for $x > 0$, where $\lambda > 0$. For $x \leq 0$, take $F_0(x) = 0$. Show that the probability density function (pdf) of X is

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}(1 - \delta + 2\delta e^{-x/\lambda})I(x > 0).$$

1.13. The rules for the game of craps are as follows. A player rolls two dice and computes the total of the spots showing.

- If the player's first toss is a 7 or 11, the player wins the game.
- If the first toss is a 2, 3, or 12, the player loses the game.
- If the player rolls anything else (4, 5, 6, 8, 9, or 10) on the first toss that value becomes the player's "point." If the player does not win or lose on the the first toss he tosses the dice repeatedly until he obtains either his point or a 7. He wins if he tosses his point before tossing a 7 and loses if he tosses a 7 before his point.

(a) A player decides to play 10 games of craps. Define the random variable X to be the number of games that s/he wins (out of 10). Find the probability mass function (pmf) and cumulative distribution function (cdf) of X and graph them side by side. Be sure to list any assumptions that you make.
(b) A stubborn player keeps playing games until he wins. Let Y denote the number of games he will play. Find the pmf and cdf of Y and graph them side by side. Be sure to list any and all assumptions that you make.

1.14. There are M different chocolate bars for Ann to choose from. The bars are labeled from 1 to M ($M \geq 4$), and the one with label i is worth i dollars, for $i = 1, 2, ..., M$.

(a) Ann chooses one chocolate bar from the M bars. Suppose the probability that Ann chooses bar i is proportional to the value of this bar, for $i = 1, 2, ..., M$. Let X denote the value of the selected chocolate bar. Provide the probability mass function of X.

(b) As in part (a), Ann chooses one chocolate bar with the probability of choosing bar i proportional to the value of this bar, for $i = 1, 2, ..., M$. Now, seeing the value of the selected bar, Ann hesitates whether or not she should get a second bar. Assume that the probability Ann selects a second bar is $(1/2)^i$, for $i = 1, 2, ..., M$ (so the cheaper the first bar is, the more likely Ann will get one more bar). Find the probability that Ann chooses a second bar.

(c) Consider a different experiment where Ann chooses three chocolate bars from this collection of M bars with replacement (i.e., after Ann chooses a bar, she places it back into the collection before choosing the next time). Assume that the outcome of one choice does not affect her other choices. Use the pmf from (a) to find the probability that the \$1 chocolate bar is chosen at least once.