From Casella and Berger, turn in the following problems from Chapter 2:

Homework 4: 3, 9, 11, 12, 14, and 23.
Homework 5: 22, 25, 28, 32, 36, and 38.

These are extra problems that I have given on past exams (in STAT 712 or in related courses). You do not have to turn these in.

2.1. Suppose X is a discrete random variable with probability mass function

$$f_X(x) = \begin{cases} (\frac{1}{2})^{x+1}, & x = 0, 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive $E(X)$ and $\text{var}(X)$. Don't just state the answers (if you know them).
(b) Derive the probability mass function of $Y = (X-2)^2$.

2.2. A continuous random variable X is said to have a *Cauchy distribution* if its probability density function is

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Define $Y = -X$ and $Z = 1/X$. Show that X, Y, and Z all have the same distribution. *Hint:* Derive the cdf of each random variable.

2.3. A random variable X is said to be *bounded* if there exists $M > 0$ such that

$$P_X(|X| \le M) = P_X(-M \le X \le M) = 1.$$

(a) Suppose X is a bounded continuous random variable with probability density function (pdf) $f_X(x)$. Prove that $E(|X|) \le M$.
(b) Suppose X is a bounded continuous random variable with pdf $f_X(x)$. Prove that the moment generating function of X exists for all $t \in \mathbb{R}$.
Hint: For both parts, you can assume that $f_X(x) = 0$ if $x \notin [-M, M]$. Explain why.

2.4. Suppose $F_X(x)$ is the cdf of a continuous random variable X. We have learned that $Y = F_X(X)$ follows a uniform distribution on $[0,1]$. What can we say about the distribution of $W = f_X(X)$, the pdf of X? Let us focus on two cases where X follows a distribution with which we are familiar. Derive the distribution of W in each case.
(a) X follows an exponential distribution with pdf given by

$$f_X(x) = \frac{1}{\beta} e^{-x/\beta} I(x > 0),$$

where $\beta > 0$.
(b) X follows the standard normal distribution with pdf given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} I(x \in \mathbb{R}).$$

2.5. Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$, that is, the pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} I(x \in \mathbb{R}),$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Let $Y = e^X$. Derive the pdf of Y and calculate $E(Y)$ and $\text{var}(Y)$. The distribution of Y is called a *lognormal distribution*.

2.6. In Chapter 3, we will study many named probability distributions. While useful in their own right, these distributions sometimes do not describe exactly what we see in real life. Sometimes statisticians "adjust" familiar distributions to construct "closer-to-reality" distributions. Suppose that X is a continuous random variable with pdf $f_X(x)$ which is symmetric about zero, that is, $f_X(x) = f_X(-x)$ for all $x \in \mathbb{R}$. However, after observing different values of X, one suspects that the true pdf of X may not be symmetric. Motivated by this new information, one adjusts $f_X(x)$ in the following way to define a new pdf that can be skewed:

$$f_X^*(x) = 2f_X(x)\pi(x),$$

where $\pi(x)$ is called a *skewing function*. The function $\pi(\cdot)$ maps $\mathbb{R}$ to $[0,1]$ and satisfies $\pi(-x) = 1 - \pi(x)$ for all $x \in \mathbb{R}$.
(a) Show that $f_X^*(x)$ is a valid pdf. *Hint:* Write the integral over $\mathbb{R}$ as the sum of two integrals. Let $u = -x$ in the first integral.
(b) Take X to have a standard normal distribution so that $E(X) = 0$ and $\text{var}(X) = 1$. Plot the pdf of X using your favorite software. Now choose two different skewing functions $\pi(x)$ and graph the corresponding revised pdfs. For each skewing function, calculate your adjusted $E(X)$ and $\text{var}(X)$. You may have to calculate these numerically.

2.7. Suppose that X follows an exponential distribution with mean $\beta = 1$, that is, the pdf of X is $f_X(x) = e^{-x}I(x > 0)$. For $-\infty < \alpha < \infty$ and $\beta > 0$, define

$$Y = g(X) = \alpha - \beta \ln X.$$

(a) Show that the pdf of Y is given by

$$f_Y(y) = \frac{1}{\beta} \exp\left\{-\left(\frac{y-\alpha}{\beta}\right) - e^{-(y-\alpha)/\beta}\right\} I(y \in \mathbb{R}).$$

The random variable Y is said to follow a *Gumbel distribution*. This distribution is useful in modeling extreme values (e.g., maximum temperature, etc.).
(b) Define

$$Z = \frac{Y-\alpha}{\beta}.$$

Show that $E(Z) = \gamma$, where γ is the *Euler-Mascheroni* constant defined via

$$\gamma = -\int_0^\infty e^{-s} \log s \, ds.$$

(c) Show that the mgf of Z in part (b) is given by $M_Z(t) = \Gamma(1-t)$, for $t < 1$, where $\Gamma(\cdot)$ is the gamma function.

(d) Derive the mgf of Y. Derive the mean and variance of Y using the mgf.

2.8. Suppose that $X_n \sim \text{gamma}(n/2, 2)$ and let

$$W_n = \frac{X_n - n}{\sqrt{2n}}.$$

(a) Derive $M_{W_n}(t)$, the mgf of W_n.
(b) Show that $M_{W_n}(t) \longrightarrow e^{t^2/2}$ for all t, as $n \to \infty$. Note that $e^{t^2/2}$ is the mgf of a standard normal random variable. Therefore, $W_n \xrightarrow{d} \mathcal{N}(0,1)$.
(c) Perform a simulation study in R to see how good the standard normal approximation is in part (b) for n large. Experiment with different values of n.

2.9. An insurance company models the number of "serious injury" claims per day, X, as a random variable with pmf

$$f_X(x) = \begin{cases} \dfrac{1}{(x+1)(x+2)}, & x = 0, 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that this is a valid pmf; that is, show that

$$\sum_{x=0}^{\infty} f_X(x) = 1.$$

(b) Show that $E(X)$ does not exist. *Hint:* Try to find a function $g(x)$ such that

$$\frac{x}{(x+1)(x+2)} > g(x),$$

for all $x \geq 1$, where the function $g(x)$ is not integrable over $[1, \infty)$; i.e., $\int_1^\infty g(x)dx = +\infty$. The result then follows from the integral comparison test from calculus.
(c) Does the mgf of X exist? If so, derive it. If not, show that it does not.

2.10. Suppose that X is a continuous random variable with pdf

$$f_X(x) = \beta^{-1} e^{-x/\beta} I(x > 0),$$

where $\beta > 0$. Derive the distribution of each of the following functions of X and derive the mean and variance of each function.
(a) $U = g_1(X) = aF_X(X) + b$, where $F_X(x)$ is the cdf of X and $a, b \in \mathbb{R}$.
(b) $V = g_2(X) = X + \theta$, where $\theta \in \mathbb{R}$.
(c) $W = g_3(X) = \cos(X)$.
(d) $Y = g_4(X) = \lceil X \rceil$, where $\lceil a \rceil$ is the *ceiling function*, that is, the smallest integer that is greater than or equal to a. For example, $\lceil 5.5 \rceil = 6$ and $\lceil 6 \rceil = 6$.
Note: I think part (c) can be done analytically, but I am not sure (note that this is not a 1:1 transformation over $\mathbb{R}^+$). If you can not make sufficient progress analytically, could you investigate this part using simulation?

2.11. In each part, derive the moment generating function (mgf) of X and then use the mgf to find $E(X)$ and $\text{var}(X)$.
(a) X has pmf

$$f_X(x) = \begin{cases} (1-\theta)\theta^x, & x = 0, 1, 2, ... \\ 0, & \text{otherwise} \end{cases}$$

where $0 < \theta < 1$.
(b) X has pdf

$$f_X(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

2.12. The random variable X has the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{2\beta} e^{-|x-\alpha|/\beta}, & -\infty < x < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

for values of $-\infty < \alpha < \infty$ and $\beta > 0$.
(a) Find $M_X(t)$, the moment generating function of X. Make sure to note for which values of t the mgf is defined.
(b) Find $E(X)$ and $\text{var}(X)$.
(c) Derive the pdf of $Y = g(X) = (X - \alpha)/\beta$.