From Casella and Berger, turn in the following problems from Chapter 3:

Homework 6: 10, 13, 14, 18, 25, and 26.
Homework 7: 24, 28, 39, 42, and 46.

These are extra problems that I have given on past exams (in STAT 712 or in related courses). You do not have to turn these in.

3.1. Suppose that X is a continuous random variable with probability density function (pdf)

$$f_X(x|\theta) = \theta x^{-(\theta+1)} I(x > 1),$$

where $\theta > 0$.
(a) Calculate an exact expression for the tail probability $P(X > c)$, for $c > 1$.
(b) Calculate the upper bound on $P(X > c)$, $c > 1$, provided by Markov's Inequality. Note any restrictions on θ that are needed for the upper bound to be applicable.

3.2. In each part below, give an example of a parametric family $\{f_X(x|\boldsymbol{\theta}); \boldsymbol{\theta} \in \Theta\}$ that satisfies the stated conditions. Specify the family, the parameter space Θ, and the support $\mathcal{X}$. Prove any claims you make.
(a) a family that is a single parameter exponential family (i.e., $d = 1$) and also a scale family.
(b) a family that is a two-parameter full exponential family (i.e., $d = k = 2$) and also a location-scale family.
(c) a two-parameter full exponential family with $\boldsymbol{\theta} = (\theta_1, \theta_2)$ that contains a subfamily, with $\theta_2 = g(\theta_1)$, that is a full one-parameter exponential family.

3.3. Recall that if X has a beta(α, β) distribution, then the probability density function (pdf) of X is

$$f_X(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1).$$

(a) Show that $\{f_X(x|\alpha, \beta); \alpha > 0, \beta > 0\}$ is a two-parameter exponential family. Is this a full or curved family?
(b) For this part only, suppose that $\alpha > 1$ and $\beta > 1$. Find the mode of X, that is, find the value of x that maximizes $f_X(x|\alpha, \beta)$.
(c) Derive the pdf of $Y = 1 - X$.

3.4. Suppose that $X \sim \mathcal{U}(a, b)$, where $a < b$.
(a) Show that X has pdf in the location-scale family generated by the standard pdf $f_Z(z) = I(0 < z < 1)$.
(b) Show that the pdf of $Y = Z(1 - Z)$ is

$$f_Y(y) = 2(1 - 4y)^{-1/2} I(0 < y < 1/4).$$

(c) Derive the pdf of $W = (X - a)(b - X)$.

3.5. Suppose $X \sim$ geometric(p), where $0 < p < 1$; i.e., the probability mass function (pmf) of

X is

$$f_X(x|p) = \begin{cases} (1-p)^{x-1}p, & x = 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that

$$P(X > t + s | X > t) = P(X > s),$$

for positive integers t and s, that is, X has the memoryless property.
(b) Show that X has pmf in the exponential family.

3.6. Suppose $X \sim$ exponential(β); i.e., the probability density function of X is

$$f_X(x|\beta) = \frac{1}{\beta} e^{-x/\beta} I(x > 0).$$

(a) Show that $\{f_X(x|\beta); \beta > 0\}$ is a scale family.
(b) Derive the moment generating function (mgf) of $Y = 2X/\beta$, making certain to note the values at which this mgf is defined.

3.7. The folded normal distribution in Problem 3.20 can be derived from the standard normal distribution.
(a) Suppose that $Z \sim \mathcal{N}(0, 1)$. Show that $X = |Z|$ has a distribution identified by the pdf in Problem 3.20.
(b) Write an R simulation to approximate the distribution of

$$Y = g(X) = \log_{10} |\sin 2\sqrt{X}|$$

and also to approximate $E(Y)$ and var(Y). In this example, you can see the value of using simulation (as deriving the pdf of Y in closed form might be difficult or impossible).

3.8. Calculate the skewness ξ and kurtosis κ for
(a) $X \sim$ Poisson(λ), $\lambda > 0$
(b) $X \sim$ gamma(α, β), $\alpha, \beta > 0$.
Discuss the different pmf/pdf shapes for each family. Note that Casella and Berger denote the skewness and kurtosis by α_3 and α_4, respectively. See Exercise 2.28 (pp 79-80).

3.9. Suppose that $X \sim$ beta(a, b), where $a > 0$ and $b > 0$. Define

$$Y = \frac{bX}{a(1-X)}.$$

(a) Derive the pdf of Y. We will see this distribution in Chapter 5.
(b) Suppose $b > 1$. Find $E(Y)$. Can this constraint on b be removed? Discuss why or why not.
(c) Show that the mgf of Y does not exist.

3.10. Suppose that Z is a continuous random variable with pdf

$$f_Z(z) = \text{sech}(\pi z) I(z \in \mathbb{R}),$$

where $\text{sech}(\cdot)$ is the hyperbolic secant function.
(a) Show that $f_Z(z)$ is a valid pdf.
(b) With $-\infty < \mu < \infty$ and $\sigma > 0$, derive the pdf of $X = \sigma Z + \mu$. Describe the collection of pdfs that arise.
(c) Pick 2 values of (μ, σ) and graph the corresponding pdfs on the same graph (one by changing the location parameter from zero and the other from changing the scale parameter from unity). Include the $(0, 1)$ case on the graph also.
(d) Calculate $E(X)$ and $\text{var}(X)$.

3.11. Suppose that X is a continuous random variable with pdf

$$f_X(x|\eta, \psi) = \exp\left\{\frac{x\eta - D(\eta)}{\psi} + c(x, \psi)\right\} I(x \in \mathcal{X}),$$

where the support $\mathcal{X}$ is free of all parameters, η and ψ are parameters, and $D(\eta)$ and $c(x, \psi)$ are real-valued functions. Assume that the conditions under which one can interchange derivatives and integrals are satisfied. Show that $E(X) = D'(\eta)$ and $\text{var}(X) = \psi D''(\eta)$, assuming these derivatives exist.