From Casella and Berger, turn in the following problems from Chapter 4:

Homework 8: 1, 4, 9, 10, 13, and 18.
Homework 9: 14, 22, 26, 31, 33, and 37.
Homework 10: 40, 45, 54, 56, 58, and 63.

These are extra problems that I have given on past exams (in STAT 712 or in related courses). You do not have to turn these in.

4.1. Suppose that $\mathbf{X} = (X_1, X_2)'$ is a bivariate random vector with pdf

$$f_{X_1,X_2}(x_1, x_2) = x_1 e^{-x_2} I(0 < x_1 < x_2 < \infty).$$

(a) Find $M_{X_1,X_2}(t_1, t_2)$, the joint moment generating function of X_1 and X_2. Does

$$M_{X_1,X_2}(t_1, t_2) = M_{X_1,X_2}(t_1, 0) M_{X_1,X_2}(0, t_2)$$

for all t_1 and t_2 where $M_{X_1,X_2}(t_1, t_2)$ exists?
(b) Find the marginal distributions of X_1 and X_2.
(c) Are X_1 and X_2 independent?
(d) Repeat parts (a), (b), and (c) with

$$f_{X_1,X_2}(x_1, x_2) = \left(\frac{1}{2}\right)^{x_1+x_2} I(x_1 = 1, 2, 3, ...,) I(x_2 = 1, 2, 3, ...,).$$

4.2. Suppose that $(X, Y)'$ is a bivariate random vector with pdf

$$f_{X,Y}(x, y) = 2\exp\{-(x + y)\} I(0 < x < y < \infty).$$

(a) Explain quickly why X and Y are dependent.
(b) Find $E(Y|X = x)$ and $\text{var}(Y|X = x)$. Your answers should depend on x.
(c) Derive the distribution of $E(Y|X)$.
(d) Derive the conditional moment generating function of Y, given $X = x$. Differentiate this mgf and evaluate it at $t = 0$; you should get $E(Y|X = x)$.

4.3. Consider the random vector $(X, Y)'$ with joint pdf

$$f_{X,Y}(x, y) = |x - y| e^{-(x+y)} I(x > 0) I(y > 0).$$

(a) Verify that this is a valid (joint) pdf.
(b) Calculate $P(0 < Y < X < 1)$.
(b) Derive the (joint) mgf of $(X, Y)'$.
(c) Are X and Y independent?

4.4. Suppose that $W_1 \sim \chi^2(\nu_1)$ and $W_2 \sim \chi^2(\nu_2)$. If W_1 and W_2 are independent, show that the pdf of

$$W = \frac{W_1/\nu_1}{W_2/\nu_2}$$

is given by

$$f_W(w) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} w^{(\nu_1-2)/2} I(w>0)}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})\left(1+\frac{\nu_1 w}{\nu_2}\right)^{(\nu_1+\nu_2)/2}}.$$

We say that W has an F *distribution* with ν_1 (numerator) and ν_2 (denominator) degrees of freedom. This is denoted by $W \sim F(\nu_1, \nu_2)$.

4.5. Suppose that X_1 and X_2 are continuous random variables with joint pdf $f_{X_1,X_2}(x_1, x_2)$, for $x_1, x_2 \in \mathbb{R}$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2$.
(a) Show that the marginal pdf of $Y = X_1 + X_2$ can be written as

$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{X_1,X_2}(y_1 - y_2, y_2) dy_2.$$

This is called the *convolution formula*. Compare this with Theorem 5.2.9 (CB, pp 215). What's the difference between these two results?
(b) Use the convolution technique to derive the pdf of $Y_1 = X_1 + X_2$ if

$$f_{X_1,X_2}(x_1, x_2) = \exp\{-(x_1 + x_2)\} I(x_1 > 0, x_2 > 0).$$

You should see that $Y_1 \sim \text{gamma}(2, 1)$.
(c) How many other ways can you prove the result in part (b)? I can immediately think of 3 other ways (maybe more?). Derive the result in part (b) in as many different ways that you can think of.

4.6. Consider the following hierarchical model:

$$\begin{aligned} X|Y &\sim \text{exponential}(1/Y) \\ Y &\sim \text{gamma}(a, b). \end{aligned}$$

(a) Find the pdf of X.
(b) Calculate $E(X)$ and $\text{var}(X)$. Note any restrictions on the values of a and b.
(c) Derive $f_{Y|X}(y|x)$ and $E(Y|X = x)$.

4.7. Suppose that X_1, X_2, X_3 are (mutually) independent and identically distributed random variables, each with common pdf $f_X(x) = e^{-x} I(x > 0)$. Define

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2} \\ Y_2 &= \frac{X_1 + X_2}{X_1 + X_2 + X_3} \end{aligned}$$

and $Y_3 = X_1 + X_2 + X_3$. Show that Y_1, Y_2, and Y_3 are mutually independent and determine each marginal distribution.

4.8. Suppose that $\mathbf{X} = (X_1, X_2)'$ is a bivariate random vector with (joint) moment-generating function $M_{X_1,X_2}(t_1, t_2)$.
(a) Show that the covariance of X_1 and X_2 can be computed using

$$\text{cov}(X_1, X_2) = \left.\frac{\partial^2 M_{X_1,X_2}(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=0,t_2=0} - \left[\left.\frac{\partial M_{X_1,X_2}(t_1,t_2)}{\partial t_1}\right|_{t_1=0,t_2=0}\right]\left[\left.\frac{\partial M_{X_1,X_2}(t_1,t_2)}{\partial t_2}\right|_{t_1=0,t_2=0}\right].$$

(b) Let $\psi(t_1, t_2) = \ln M_{X_1,X_2}(t_1, t_2)$. Show that

$$\text{cov}(X_1, X_2) = \left.\frac{\partial^2 \psi(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=0,t_2=0}.$$

(c) Suppose that $\mathbf{X} = (X_1, X_2)'$ has a bivariate normal distribution with marginal means μ_1 and μ_2, marginal variances σ_1^2 and σ_2^2, and correlation ρ. The moment-generating function of $\mathbf{X}$ is

$$M_{\mathbf{X}}(\mathbf{t}) = \exp(\mathbf{t}'\boldsymbol{\mu} + \mathbf{t}'\mathbf{V}\mathbf{t}/2),$$

where $\mathbf{t} = (t_1, t_2)'$, $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, and

$$\mathbf{V} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix},$$

where $\sigma_{12} = \sigma_{21} = \text{cov}(X_1, X_2)$. Show that, in fact, one does get $\text{cov}(X_1, X_2) = \sigma_{12}$ using the result in part (b).

4.9. Suppose that X_1 and X_2 are independent $\mathcal{U}(0, 1)$ random variables.
(a) Derive the pdf of $\overline{X} = (X_1 + X_2)/2$.
(b) Calculate

$$E\left(\frac{X_1}{\overline{X}}\right).$$

(c) Calculate $E(X_1|\overline{X})$, $\text{var}(X_1|\overline{X})$, and $\text{cov}(X_1, \overline{X})$.

4.10. Consider the random vector $(X, Y)'$ with probability density function (pdf)

$$f_{X,Y}(x, y) = 2x^{-1}e^{-2x}I(0 < y < x < \infty).$$

(a) Verify that this is a valid pdf.
(b) Calculate $P(Y > X/2, X < 1)$.
(c) Calculate the correlation of X and Y.

4.11. Suppose that $(X, Y)'$ has the following probability density function (pdf)

$$f_{X,Y}(x, y) = (xy)^{-2}I(x > 1)I(y > 1).$$

(a) Are X and Y independent? Explain.
(b) Find the joint pdf of $U = XY$ and $V = X/Y$.
(c) Find the marginal pdfs of U and V. Are U and V independent?

4.12. Suppose that $X_1, X_2, ..., X_n$ are mutually independent random variables. Each random variable, marginally, follows a standard normal distribution.
(a) Derive the distribution of $S = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$.
(b) Derive the distribution of $T = X_1^2 + X_2^2 + \cdots + X_n^2$.
(c) Derive the moment-generating functions (mgf) of

$$U = \sqrt{n}S \qquad \text{and} \qquad V = \frac{T-n}{\sqrt{2n}}.$$

(Derive each one separately; I am not asking for a "joint" mgf here). What does each mgf converge to as $n \to \infty$?

4.13. Suppose that X and Y are random variables with finite means and variances. Suppose that we want to predict Y as a linear function of X. That is, we are interested only in functions of the form $Y = \beta_0 + \beta_1 X$, for fixed constants β_0 and β_1. Define the *mean squared error of prediction* by

$$Q(\beta_0, \beta_1) \equiv E\{[Y - (\beta_0 + \beta_1 X)]^2\}.$$

(a) Show that $Q(\beta_0, \beta_1)$ is minimized when

$$\beta_1 = \rho \left(\frac{\sigma_Y}{\sigma_X} \right)$$

and

$$\beta_0 = E(Y) - \beta_1 E(X),$$

where $\rho = \text{corr}(X, Y)$.
(b) Calculate the values of β_0 and β_1 for the bivariate distribution in (extra) Problem 4.10. Does $\beta_0 + \beta_1 X$ equal $E(Y|X)$ in that problem? Comment.

4.14. A philanthropist decides to choose s persons at random and to invite them to a series of parties over the year. A party is given on a particular day if at least one person (among the s) has a birthday on that day (multiple parties cannot occur on the same day). Assume, for simplicity, that there are exactly 365 days in the year and that birthdays occur at random across days. Define

$$X_i = \begin{cases} 1, & \text{if a party is given on day } i, \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, ..., 365$. Therefore, $X_1, X_2, ..., X_{365}$ is a sequence of 0-1 random variables.
(a) What is the distribution of X_1?
(b) Compute $\text{cov}(X_i, X_j)$, where $i \neq j$.
(c) Let T denote the number of parties that will be given over the year. Find the mean and variance of T.

4.15. Suppose that U_1 and U_2 are independent random variables with $U_1 \sim \chi^2_{n_1}(\lambda)$ and $U_2 \sim \chi^2_{n_2}$. That is, U_1 is distributed as noncentral χ^2 with degrees of freedom $n_1 > 0$ and noncentrality parameter $\lambda > 0$, and U_2 is distributed as (central) χ^2 with degrees of freedom $n_2 > 0$.
(a) Recall that the distribution of U_1 arises as a mixture in the following hierarchy:

$$\begin{aligned} U_1|Y &\sim \chi^2_{n_1+2Y} \\ Y &\sim \text{Poisson}(\lambda). \end{aligned}$$

Show that $E(U_1) = n_1 + 2\lambda$ and $\text{var}(U_1) = 2n_1 + 8\lambda$.
(b) We know the random variable

$$W = \frac{U_1/n_1}{U_2/n_2}$$

has a non-central F distribution, seen in Chapter 5. Show that

$$E(W) = \frac{n_2}{n_2 - 2}\left(1 + \frac{2\lambda}{n_1}\right)$$

for $n_2 > 2$. Make sure you explain why $n_2 > 2$.

4.16. Let $F_X : \mathbb{R} \to [0,1]$ and $F_Y : \mathbb{R} \to [0,1]$ be univariate cumulative distribution functions (cdfs) and suppose $-1 \leq \alpha \leq 1$. Define $F^{(\alpha)}_{X,Y} : \mathbb{R}^2 \to [0,1]$ by

$$F^{(\alpha)}_{X,Y}(x,y) = F_X(x)F_Y(y)\{1 + \alpha[1 - F_X(x)][1 - F_Y(y)]\}.$$

The collection $\{F^{(\alpha)}_{X,Y} : -1 \leq \alpha \leq 1\}$ is called the *Farlie-Morgenstern family* of bivariate cdfs corresponding to F_X and F_Y.
(a) Starting with the expression for $F^{(\alpha)}_{X,Y}(x,y)$, show that the marginal cdfs of X and Y are given by $F_X(x)$ and $F_Y(y)$, respectively.
(b) What value of α corresponds to X and Y being independent? Explain.
(c) As a special case, suppose that X and Y are each distributed as exponential with mean 1. In this case, for any $\alpha \in [-1,1]$, show that the joint probability density function of $(X,Y)'$ is

$$f^{(\alpha)}_{X,Y}(x,y) = \{1 + \alpha[(1 - 2e^{-x})(1 - 2e^{-y})]\}e^{-(x+y)}I(x > 0)I(y > 0).$$

4.17. Let α be a positive constant. Let $(X,Y)'$ be a bivariate random vector that is uniformly distributed on the circle with center $(0,0)$ and radius α. That is, the probability density function (pdf) of $(X,Y)'$ is

$$f_{X,Y}(x,y) = \frac{1}{\pi\alpha^2}\, I(x^2 + y^2 < \alpha^2).$$

Define $R = \sqrt{X^2 + Y^2}$ and $S = Y/X$.
(a) Derive the cumulative distribution function (cdf) of R.
(b) Find the pdf of R.
(c) Show that R and S are independent.
(d) What is the geometrical interpretation of the two variables R and S?

4.18. Suppose the random vector $(X,Y)'$ has probability density function:

$$f_{X,Y}(x,y) = \frac{e^{-x/y}e^{-y}}{y} I(x>0, y>0).$$

(a) Find $E(X|Y=y)$.
(b) Find $\text{corr}(X,Y)$.

4.19. Let X denote the concentration of a certain substance based on one trial of an experiment, and let Y denote the concentration of the substance based on a second trial. Assume that the joint probability density function of (X,Y) is given by $f_{X,Y}(x,y) = 4xy$, if $0 < x < 1$ and $0 < y < 1$, and $f_{X,Y}(x,y) = 0$ otherwise.
(a) Derive the joint cumulative distribution function of (X,Y).
(b) Find the probability that the average concentration from the two trials is less than 0.5.
(c) Find the probability that the concentration from the first trial is less than 0.5.

4.20. Suppose that

$$X|Y=y \sim \text{binomial}(y,p),$$

where Y denotes the number of trials in a binomial experiment and p is the probability of "success" for each trial. Assume that, marginally,

$$Y \sim \text{nib}(r,\pi),$$

where Y is defined as the number of failures before the rth success, and π is the probability of "success" for each trial in the negative binomial experiment. Show that, marginally,

$$X \sim \text{nib}(r, \pi/\{1-(1-p)(1-\pi)\}).$$

That is, show that X records the number of failures before the rth success and the (marginal) probability of success is $\pi/\{1-(1-p)(1-\pi)\}$. *Hint:* One can try to derive the marginal probability mass function of X (this should be doable). You could also try to derive the marginal moment generating function (mgf) of X. Since the mgf is an expectation, you could use the iterated expectation technique to avoid heavy algebra.

4.21. Suppose that X is a random variable with $E(X) = -1$, $\text{var}(X) = 1$ and $P(X < 0) = 1$. Argue that $E(X^3) < -1$. You must argue that the inequality is strict.