GROUND RULES:

- This exam contains 6 questions. The questions are of equal weight.
- Print your name at the top of this page in the upper right hand corner.
- This exam is closed book and closed notes.
- Show all of your work and explain all of your reasoning! **Translation:** No work, no credit. Insufficient explanation, no credit. If you are unsure about whether or not you should explain a result or step in your derivation/proof, then this means you probably should explain it.
- Do not talk with anyone else about this exam. You must work by yourself. No communication of any type with others.

1. Suppose the continuous random variable X has probability density function

$$f_X(x|\mu, \sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x - \mu|}{\sigma} \right), \quad x \in \mathbb{R},$$

where $-\infty < \mu < \infty$ and $\sigma > 0$.

(a) Show that $\{f_X(x|\mu, \sigma); -\infty < \mu < \infty,\ \sigma > 0\}$ is a location-scale family and identify the standard density.

(b) Does X have pdf in the exponential family? Prove any claims you make.

(c) For $r > 0$, calculate

$$P\left(\frac{|X - \mu|}{\sigma} > r \right)$$

exactly. Compare this calculation with Markov's upper bound for this probability.

2. Suppose X has a geometric distribution; i.e., the probability mass function (pmf) of X is

$$f_X(x|p) = \begin{cases} (1-p)^{x-1}p, & x = 1, 2, 3, ..., \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < p < 1$.

(a) Show that $\{f_X(x|p), 0 < p < 1\}$ is a one-parameter exponential family.

(b) Derive the moment-generating function of X.

(c) For this part only, suppose that, conditional on $P = p$, the random variable X follows a geometric distribution. In other words, the first level of the corresponding hierarchical model is

$$X|P = p \sim f_{X|P}(x|p),$$

where the conditional pmf, now written $f_{X|P}(x|p)$, is given above. The second level of the hierarchical model is $P \sim \text{beta}(a, b)$, where $a > 1$ is known and $b = 1$. Derive the marginal pmf of X and find $E(X)$ under this hierarchical model.

(d) Describe a real application where the hierarchical model in part (c) might be useful.

3. Consider the random vector (X, Y) with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} c(1+xy), & -1 < x < 1, -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $c = 1/4$.

(b) Calculate $P(Y > X^2)$.

(c) Derive the marginal probability density functions of X and Y. Show that X and Y are not independent.

(d) Even though X and Y are not independent, it turns out that X^2 and Y^2 are independent! Prove this. *Hint*: Work with $P(X^2 \leq u, Y^2 \leq v)$, the joint cdf of (X^2, Y^2).

4. Consider the random vector (X, Y) with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $E(Y)$ does not exist but that $E(Y|X = x) = 1/x$.

(b) Find the variance of $E(X|Y)$.

5. Suppose X_1, X_2, and X_3 are random variables with joint probability density function (pdf)

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = 6e^{-x_1-x_2-x_3} I(0 < x_1 < x_2 < x_3 < \infty).$$

Define the random variables

$$U_1 = X_1, \quad U_2 = X_2 - X_1, \quad \text{and} \quad U_3 = X_3 - X_2.$$

(a) Derive the joint pdf of $\mathbf{U} = (U_1, U_2, U_3)$. Identify the marginal distributions of U_1, U_2, and U_3 separately.

(b) Find the mean and variance of $W = -2U_1 + 3U_2 - U_3$.

6. Suppose $X_1, X_2, ..., X_n$ are mutually independent $b(m_i, p_i)$ random variables; i.e., the probability mass function of X_i is given by

$$f_{X_i}(x_i|p_i) = \binom{m_i}{x_i} p_i^{x_i}(1-p_i)^{m_i - x_i}, \quad x_i = 0, 1, 2, ..., m_i,$$

for $i = 1, 2, ..., n$. The success probabilities $0 < p_i < 1$ are unknown parameters. The number of trials m_i are fixed and known.

(a) If $p_1 = p_2 = \cdots = p_n = p$, derive the distribution of $Z = \sum_{i=1}^{n} X_i$. What can you say about the distribution of Z is the p_i's are different?

(b) Suppose the success probability associated with X_i is

$$p_i = \frac{\exp(a + bw_i)}{1 + \exp(a + bw_i)}, \quad i = 1, 2, ..., n,$$

where the w_i's are fixed constants and $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are unknown parameters. Show that the pmf of X_i is a two-parameter exponential family.

(c) For this part only, suppose $n = 2$ and $p_1 = p_2 = p$. Derive the (conditional) probability mass function of X_1 given $Z = X_1 + X_2 = r$, where $r > 0$.