1. For $0 < p < 1$, recall that the $p$th quantile of a continuous random variable $X$, denoted by $\phi_p$, satisfies

$$F_X(\phi_p) = p,$$

where $F_X(\cdot)$ is the cumulative distribution function (cdf) of $X$. This definition assumes that the cdf is a strictly increasing function (i.e., there are no flat regions) over the support of $X$. In this problem, we will assume that $X_1, X_2, ..., X_n$ are iid exponential($\beta$), where $\beta > 0$ is unknown.
(a) Derive the population cdf of $X \sim$ exponential($\beta$).
(b) On the basis of observing $X_1, X_2, ..., X_n$ (an iid exponential sample), find the maximum likelihood estimator of $\phi_p$. Denote your estimator by $\widehat{\phi}_p$. *Hint:* First show $\phi_p = -\beta \ln(1-p)$.
(c) Under the exponential assumption, find the asymptotic distribution of $\widehat{\phi}_p$, properly centered and scaled.
(d) Use your asymptotic distribution in part (c) to derive a large sample $1-\alpha$ confidence set for $\phi_p$.

2. Suppose $X_1, X_2, ..., X_n$ are iid from $f_X(x|\theta) = \theta x^{\theta-1} I(0 < x < 1)$, where $\theta > 0$ is unknown. Note that the population distribution here is beta($\theta, 1$).
(a) Derive the likelihood ratio test of

$$\begin{gathered} H_0 : \theta = 1 \\ \text{versus} \\ H_1 : \theta \neq 1. \end{gathered}$$

Express your decision rule in terms of a sufficient statistic and show how to obtain the critical value for the test.
(b) Invert the acceptance region of your test in part (a) and write a $1-\alpha$ confidence set for $\theta$.
(c) Consider the same model setup as above, but now put the following prior distribution on $\theta$:

$$\theta \sim \pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} I(\theta > 0),$$

where $a > 0$ and $b > 0$ are known. Show how to find a $1-\alpha$ credible set for $\theta$.

3. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = 2\theta x e^{-\theta x^2} I(x > 0).$$

where $\theta > 0$ is unknown.
(a) Find the uniformly minimum variance unbiased estimator (UMVUE) of $\theta$.
(b) Find the uniformly most powerful (UMP) test of

$$\begin{gathered} H_0 : \theta \leq \theta_0 \\ \text{versus} \\ H_1 : \theta > \theta_0, \end{gathered}$$

where $\theta_0$ is known.
(c) Derive an expression for the power function of the test in part (b).

4. Suppose $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(\theta, a\theta^2)$, where $a > 0$ and $-\infty < \theta < \infty$. Both $a$ and $\theta$ are unknown. Assume $\theta \neq 0$ for obvious reasons.
(a) Find method of moments estimators of $\theta$ and $a$.
(b) Show that $\widehat{\theta} = \overline{X}$ and

$$\widehat{a} = \frac{S_b^2}{\overline{X}^2}$$

solve the score equations. Recall that $S_b^2$ is the sample variance with $n$ in the denominator (not $n-1$).
(c) Argue that $\widehat{a} \xrightarrow{p} a$, as $n \to \infty$. Then, use your intuition (and perhaps some mathematics) to formulate a sensible large sample test of

$$H_0 : a = 1$$
versus
$$H_1 : a \neq 1.$$

A good answer need not be overly mathematical.

5. In sampling land areas for counts of an animal species, we obtain an iid sample of counts $X_1, X_2, ..., X_n$, where each $X_i$ has a Poisson distribution with mean $\lambda > 0$.
(a) Show that maximum likelihood estimator (MLE) of $\lambda$, on the basis of observing $X_1, X_2, ..., X_n$, is

$$\widehat{\lambda} = \overline{X}.$$

(b) For simplicity in sampling, sometimes only the presence/absence of a species is recorded. That is, for each area, an investigator will record

$$Y_i = I(X_i \geq 1).$$

For example, if $X_1 = 1$, then $Y_1 = 1$. If $X_2 = 3$, then $Y_2 = 1$. If $X_3 = 0$, then $Y_3 = 0$. Let $T$ denote the number of $X_i$'s (and hence $Y_i$'s) that are zero.

- Write down the binomial likelihood based only on observing $T = t$.
- Show that the MLE of $\lambda$ based on observing $T$ only is

$$\widetilde{\lambda} = -\ln\left(\frac{T}{n}\right).$$

(c) Recall that the asymptotic relative efficiency (ARE) of $\widehat{\lambda}$ to $\widetilde{\lambda}$ is given by

$$\text{ARE}(\widehat{\lambda} \text{ to } \widetilde{\lambda}) = \frac{\sigma^2_{\widetilde{\lambda}}}{\sigma^2_{\widehat{\lambda}}},$$

where $\sigma^2_{\widehat{\lambda}}$ and $\sigma^2_{\widetilde{\lambda}}$ are the variances of the (suitably centered and scaled) asymptotic distributions of $\widehat{\lambda}$ to $\widetilde{\lambda}$, respectively. Show that

$$\text{ARE}(\widehat{\lambda} \text{ to } \widetilde{\lambda}) = \frac{\lambda}{e^{\lambda} - 1}.$$

(d) The overall goal is to estimate $\lambda$. Comment on the possible use of $\widetilde{\lambda}$ instead of $\widehat{\lambda}$. Would you ever advise an investigator to do this? If so, when?