1. Suppose $X \sim \text{Poisson}(\mu)$, where $\mu > 0$. Define

$$Y = \frac{X - \mu}{\sqrt{\mu}}.$$

(a) Show that the moment generating function of Y is

$$M_Y(t) = E(e^{tY}) = \exp\{\mu e^{t/\sqrt{\mu}} - \mu - t\sqrt{\mu}\}.$$

(b) Show that you can write

$$\mu e^{t/\sqrt{\mu}} = \mu + t\sqrt{\mu} + \frac{t^2}{2} + o(1),$$

where $o(1)$ is a term that satisfies $o(1) \to 0$, as $\mu \to \infty$. *Hint:* Expand $e^{t/\sqrt{\mu}}$ in a Taylor series about $t = 0$.
(c) Combine parts (a) and (b) and show that

$$Y = \frac{X - \mu}{\sqrt{\mu}} \xrightarrow{d} \mathcal{N}(0, 1),$$

as $\mu \to \infty$.

2. Suppose $X_1, X_2, ..., X_n$ are iid $\mathcal{U}(0, \theta)$, where $\theta > 0$. In class, we derived the method of moments and maximum likelihood estimators of θ to be, respectively,

$$\begin{aligned} \widehat{\theta}_1 &= 2\overline{X} \\ \widehat{\theta}_2 &= X_{(n)}. \end{aligned}$$

(a) Prove that, as $n \to \infty$,

$$\sqrt{n}(\widehat{\theta}_1 - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2/3).$$

(b) Prove that, as $n \to \infty$,

$$Y_n = n(\theta - \widehat{\theta}_2) \xrightarrow{d} Y,$$

where Y is an exponential random variable with mean θ. *Hint:* Show that the cumulative distribution function (cdf) of Y_n converges pointwise to the cdf of Y.

3. Suppose $X_1, X_2, ..., X_n$ are iid random variables with population distribution

$$f_X(x|p) = \begin{cases} (1-p)^{x-1}p, & x = 1, 2, 3, ..., \\ 0, & \text{otherwise}, \end{cases}$$

where $0 < p < 1$. Note that this is a geometric distribution that describes the number of Bernoulli trials needed to find the first "success." Recall that $E(X) = 1/p$ and $\text{var}(X) = 1/p^2$.
(a) The statistic $\overline{X}$ converges in probability to a constant, as $n \to \infty$. Explain why and then identify the constant.
(b) Carefully argue that

$$Z_n = \frac{\overline{X} - \frac{1}{p}}{\overline{X}/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1),$$

as $n \to \infty$.

4. Suppose $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(\mu, 1)$, where $-\infty < \mu < \infty$. Set $\mathbf{X} = (X_1, X_2, ..., X_n)$. Define the statistic

$$T = T(\mathbf{X}) = \sum_{i=1}^{n} X_i.$$

(a) Using the definition of sufficiency, prove that T is a sufficient statistic. Do not the Factorization Theorem or exponential family theory (for this part only).
(b) Is T a complete statistic? Prove or disprove.

5. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|a,\theta) = \begin{cases} \dfrac{a}{\theta^a} x^{a-1}, & 0 < x < \theta \\ 0, & \text{otherwise,} \end{cases}$$

where $a > 0$ and $\theta > 0$ are both unknown parameters. Carefully argue that $X_{(n)}$ and

$$\frac{X_{(1)}}{\sum_{i=1}^{n} i X_{(i)}}$$

are independent. *Here is how to get started:* Fix $a = a_0$ and show that the resulting subfamily $f_X(x|\theta)$ is a scale family.

6. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\mu) = \begin{cases} e^{-(x-\mu)}, & x \geq \mu \\ 0, & \text{otherwise,} \end{cases}$$

where $-\infty < \mu < \infty$.
(a) Find the method of moments estimator of μ.
(b) Find the maximum likelihood estimator (MLE) of μ.
(c) Find the MLE of μ subject to the restriction that $\mu \leq 0$.