1. Suppose $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$ and both parameters are unknown. Set $\mathbf{X} = (X_1, X_2, ..., X_n)$ and $\boldsymbol{\theta} = (\mu, \sigma^2)$.
(a) Show that

$$\mathbf{T} = \mathbf{T}(\mathbf{X}) = \begin{pmatrix} \overline{X} \\ S^2 \end{pmatrix}$$

is a minimal sufficient statistic for $\boldsymbol{\theta}$.
(b) Define

$$W = \left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)S^2}{\sigma^2}.$$

Find the finite-sample distribution of W.
(c) Is W an ancillary statistic? Explain.

2. Recall that if X has a beta(α, β) distribution, then the probability density function (pdf) of X is

$$f_X(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1).$$

where $\alpha > 0$ and $\beta > 0$. In this problem, we are going to consider the beta subfamily where $\alpha = \beta = \theta$. Let $X_1, X_2, ..., X_n$ denote an iid sample from a beta(θ, θ) distribution.
(a) Show that

$$T(\mathbf{X}) = \prod_{i=1}^{n} X_i(1 - X_i)$$

is a sufficient statistic for θ. Is $T(\mathbf{X})$ complete?
(b) The two-dimensional statistic

$$\mathbf{T}^* = \mathbf{T}^*(\mathbf{X}) = \begin{pmatrix} \displaystyle\prod_{i=1}^{n} X_i \\ \displaystyle\prod_{i=1}^{n} (1 - X_i) \end{pmatrix}$$

is also a sufficient statistic for θ. What must be true about the conditional distribution $f_{\mathbf{X}|\mathbf{T}^*}(\mathbf{x}|\mathbf{t})$?
(c) Show that $\mathbf{T}^*(\mathbf{X})$ is not a complete statistic.

3. Suppose $W_n \sim \chi^2_n$.
(a) Prove $W_n \stackrel{d}{=} X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, ..., X_n$ are iid χ^2_1. Recall that "$\stackrel{d}{=}$" means "equal in distribution."
(b) Explain why $W_n/n \xrightarrow{p} 1$, as $n \to \infty$.
(c) Find sequences of constants a_n, b_n, and c_n such that

$$a_n(W_n/n - b_n) + c_n \xrightarrow{d} \mathcal{N}(-3, 5),$$

as $n \to \infty$. Prove your answer. **Note:** Sequences of constants can be constant sequences of constants.
(d) Suppose $V_n \sim F_{n,2n}$. Does V_n converge in probability to 1/2, 1, or 2? Or, does V_n not converge at all? Prove your claim.

4. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \frac{2x}{\theta} e^{-x^2/\theta} I(x > 0),$$

where $\theta > 0$ is unknown.
(a) Argue that $e^{-\overline{X}_n}$ converges in probability to a constant and find the constant.
(b) Define the sequence

$$\widehat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

Prove that $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2)$, as $n \to \infty$.
(c) Find a function of the $\widehat{\theta}_n$ sequence, say $g(\widehat{\theta}_n)$, whose large-sample variance is free of θ. *Hint:* Use the (first-order) Delta Method.

5. Suppose $X_1, X_2, ..., X_n$ are iid exponential with mean $\theta > 0$.
(a) Prove that

$$T = \sum_{i=1}^{n} X_i \quad \text{and} \quad S = \frac{nX_{(1)}}{\sum_{i=1}^{n} X_i}$$

are independent statistics.
(b) What is the (finite-sample) sampling distribution of T?
(c) Calculate $E(S)$.

6. Suppose that Y is a discrete random variable with probability mass function (pmf)

$$\begin{aligned} P(Y = 0) &= p + (1-p)e^{-\lambda} \\ P(Y = y) &= (1-p)\frac{\lambda^y e^{-\lambda}}{y!}, \quad y = 1, 2, 3, ... \end{aligned}$$

(a) Reparameterize the model by defining $\pi = P(Y = 0) = p + (1-p)e^{-\lambda}$. Solve this last equation for p in terms of π and λ and then substitute so that the pmf of Y depends only on π and λ. Show your work.
(b) For an iid sample $Y_1, Y_2, ..., Y_n$, let n_0 denote the number of zeroes in the sample. Show that the likelihood $L(\pi, \lambda|\mathbf{y})$ factors into two pieces and that the maximum likelihood estimator of π is $\widehat{\pi} = n_0/n$. Also, show that the maximum likelihood estimator for λ is the solution to a simple nonlinear equation involving $\overline{Y}_+$, the average of the nonzero Y values.
(c) Under the reparameterized model, set up equations that, when solved, will produce the method of moments estimators of π and λ. You do not have to solve the equations.