1. Suppose $X_1, X_2, ..., X_n$ are iid from

$$f_X(x|\theta) = \begin{cases} \theta^2 x e^{-\theta x}, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$.
(a) Find the method of moments (MOM) estimator of θ.
(b) Find the maximum likelihood estimator (MLE) of θ.
(c) Find the MLE of $E_\theta(X)$.
(d) Is there a function of θ, say $\tau(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao Lower Bound? If so, find it and identify the corresponding estimator. If not, show why not.

2. Suppose that $X_1, X_2, ..., X_n$ are iid from

$$f_X(x|\theta) = \begin{cases} (1-\theta)^{x-1}\theta, & x = 1, 2, 3, ..., \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < \theta < 1$. Note that this is a geometric (population) distribution with $E_\theta(X) = 1/\theta$.
(a) Show that $T = T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a sufficient statistic for θ and derive its (finite-sample) sampling distribution.
(b) Now view the X's as iid from $f_X(x|\theta)$ but conditionally on θ, where $\theta \sim \text{beta}(a, b)$, a, b known. Derive the posterior distribution of θ given $\mathbf{X} = \mathbf{x}$.
(c) Find $\widehat{\theta}_B = E(\theta|\mathbf{X} = \mathbf{x})$, the posterior mean of θ.

3. Suppose $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(\theta, 1)$, where $-\infty < \theta < \infty$.
(a) Show that $W(\mathbf{X}) = \overline{X}$ is the uniformly minimum variance unbiased estimator (UMVUE) for θ in **two ways**: one way that uses a Cramér-Rao Lower Bound argument and one way that uses sufficiency and completeness.
(b) Find the UMVUE for $\tau(\theta) = e^\theta$.
(c) Calculate $E(X_1|\overline{X})$, $E(X_1 - \overline{X}|\overline{X})$, and $\text{cov}(X_1, X_1 - \overline{X}|\overline{X})$.

4. Suppose $X_1, X_2, ..., X_m$ are iid $\mathcal{N}(0, \sigma_1^2)$. Suppose $Y_1, Y_2, ..., Y_n$ are iid $\mathcal{N}(0, \sigma_2^2)$. Suppose the samples are independent.
(a) Derive the likelihood ratio test (LRT) statistic $\lambda(\mathbf{x}, \mathbf{y})$ for testing

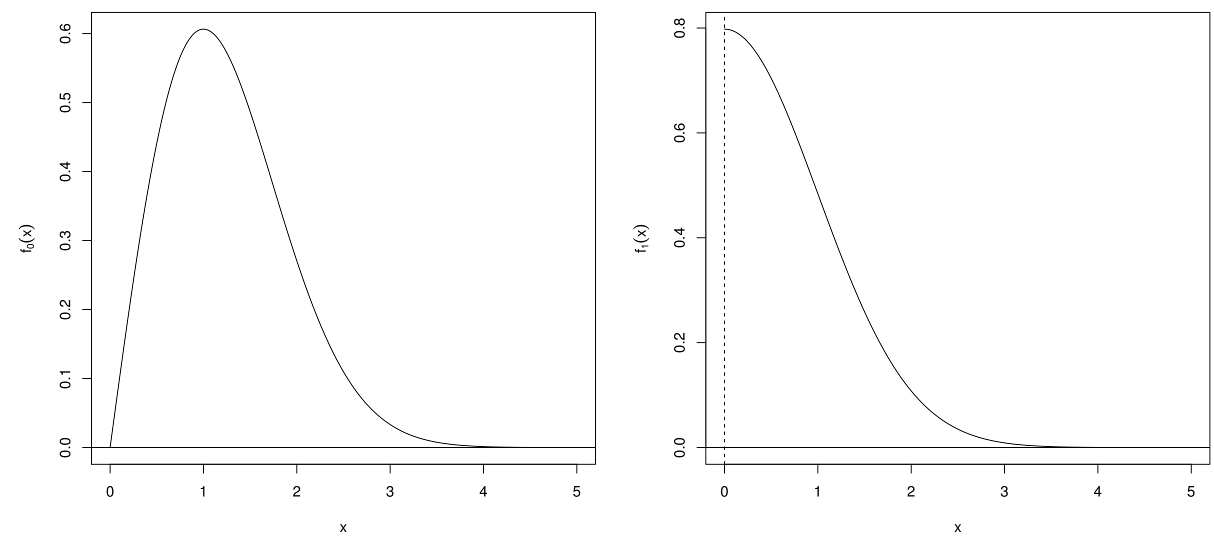
$$H_0 : \sigma_1^2 = \sigma_2^2$$
$$\text{versus}$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2,$$

and show that it is a function of $t_1 = t_1(\mathbf{x}) = \sum_{i=1}^m x_i^2$ and $t_2 = t_2(\mathbf{y}) = \sum_{j=1}^n y_j^2$.
(b) Show how you could perform a size α test in part (a) using the F distribution.

5. The random variable X has two possible distributions: $f_0(x) = xe^{-x^2/2}I(x > 0)$ or

$$f_1(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2} I(x > 0).$$

(a) Find the most powerful level $\alpha = 0.05$ test of $H_0 : X \sim f_0(x)$ versus $H_1 : X \sim f_1(x)$ on the basis of observing X only.
(b) Calculate the power of your test in part (a).
(c) Here is what the $f_0(x)$ and $f_1(x)$ densities look like:



For this part only, suppose you have an iid sample $X_1, X_2, ..., X_n$ from

$$f_X(x|p) = pf_0(x) + (1-p)f_1(x),$$

where $0 \leq p \leq 1$. Suggest a sensible test function $\phi(\mathbf{x})$ you could use to test $H_0 : p = 0.5$ versus $H_1 : p \neq 0.5$. You don't have to do anything formal here; you could just use your intuition (of course, explain your intuition!).

6. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \frac{1}{\theta} x^{(1-\theta)/\theta} I(0 < x < 1),$$

where $\theta > 0$.
(a) Derive the uniformly most powerful (UMP) level α test for

$$H_0 : \theta \leq \theta_0$$
versus
$$H_1 : \theta > \theta_0.$$

You should be able to write your rejection region in terms of gamma or χ^2 quantiles.
(b) Derive an expression for $\beta(\theta)$, the power function of the test in part (a).