GROUND RULES:

- This exam contains 5 questions. The questions are of equal weight.
- Print your name at the top of this page in the upper right hand corner.
- This exam is closed book and closed notes.
- Show all of your work and explain all of your reasoning! **Translation:** No work, no credit. Insufficient explanation, no credit. If you are unsure about whether or not you should explain a result or step in your derivation/proof, then this means you probably should explain it.
- Do not talk with anyone else about this exam. You must work by yourself. No communication of any type with others.

1. An ecologist observes data (x_i, Y_i), $i = 1, 2, ..., n$, where $x_i > 0$ is the size of an area and Y_i is the number of moss plants in the area. Suppose the Y_i's are mutually independent with $Y_i \sim \text{Poisson}(\beta x_i)$, where $\beta > 0$ is unknown. The x_i's are regarded as known constants measured without error.

(a) Derive the maximum likelihood estimator (MLE) of β. Derive its mean and variance.

(b) The least squares estimator (LSE) is the value of β that minimizes

$$Q(\beta) = \sum_{i=1}^{n} (Y_i - \beta x_i)^2.$$

Find the LSE of β. Derive its mean and variance.

(c) Derive the Crámer-Rao Lower Bound on the variance of unbiased estimators of β. Does the MLE's variance or LSE's variance attain this lower bound? If not, find an estimator that does.

2. Suppose $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 > 0$ is unknown.

(a) Consider testing $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$. The test that rejects H_0 when

$$\sum_{i=1}^{n} X_i^2 \geq \sigma_0^2 \chi^2_{n,\alpha/2} \quad \text{or} \quad \sum_{i=1}^{n} X_i^2 \leq \sigma_0^2 \chi^2_{n,1-\alpha/2}$$

where $\chi^2_{n,\alpha/2}$ ($\chi^2_{n,1-\alpha/2}$) is the upper (lower) $\alpha/2$ quantile of the χ^2 distribution with n degrees of freedom, is a uniformly most powerful unbiased (UMPU) test. In a few sentences, explain what a UMPU test is and why it is a useful concept here.

(b) Invert the acceptance region of the UMPU test in part (a) to write a $1 - \alpha$ confidence set for σ^2.

(c) Show that $\widehat{\sigma}^2$, the maximum likelihood estimator (MLE) of σ^2, satisfies

$$\sqrt{n}(\widehat{\sigma}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}(0, 2\sigma^4),$$

as $n \to \infty$.

(d) Find a consistent estimator of the asymptotic variance in part (c).

Note: In parts (c) and (d), the regularity conditions needed for the MLE $\widehat{\sigma}^2$ to be CAN hold.

3. Suppose $X_1, X_2, ..., X_n$ are iid from

$$f_X(x|\theta) = \begin{cases} \dfrac{x}{\theta} e^{-x^2/2\theta}, & x > 0 \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta > 0$.

(a) Show that $\{f_X(x|\theta) : \theta > 0\}$ is a scale family. Identify the standard density $f_Z(z)$ and the scale parameter.

(b) Find a complete and sufficient statistic $T = T(\mathbf{X})$.

(c) Find the UMVUE of $E_\theta(X)$.

(d) Let $W = W(\mathbf{X}) = \sum_{i=1}^{k} X_i^2 / \sum_{i=1}^{n} X_i^2$, for $k < n$. Show the conditional distribution $f_{T|W}(t|w)$ does not depend on k.

4. Suppose $X_1, X_2, ..., X_n$ are iid from $f_X(x|p,\theta)$, where

$$f_X(x|p,\theta) = \begin{cases} \frac{p}{\theta}e^{-x/\theta}, & x > 0 \\ \frac{1-p}{\theta}e^{x/\theta}, & x \leq 0, \end{cases}$$

where $0 < p < 1$ and $\theta > 0$. Both parameters are unknown.

(a) I have calculated $E(X) = (2p-1)\theta$ and $E(X^2) = 2\theta^2$. Find the method of moments estimators of p and θ.

(b) Show that $(D, K)'$ is a sufficient statistic, where

$$\begin{aligned} D &= \sum_{i=1}^{n} I(X_i > 0) \\ K &= \sum_{i=1}^{n} X_i I(X_i > 0) - \sum_{i=1}^{n} X_i I(X_i \leq 0). \end{aligned}$$

That is, D is the number of positive X_i's, and K is the sum of the positive X_i's minus the sum of the negative X_i's. *Hint:* Show that $f_X(x|p,\theta)$ can be written as

$$f_X(x|p,\theta) = \left(\frac{p}{\theta}e^{-x/\theta}\right)^{I(x>0)} \left(\frac{1-p}{\theta}e^{x/\theta}\right)^{1-I(x>0)}.$$

Now find $f_{\mathbf{X}}(\mathbf{x}|p,\theta)$ and use the Factorization Theorem.

(c) Show that $(D/n, K/n)'$ maximizes the likelihood function of p and θ. Don't worry about verifying second order conditions.

(d) The information matrix $\mathbb{I}(p,\theta)$ based on 1 observation is

$$\mathbb{I}(p,\theta) = -E\begin{pmatrix} \frac{\partial^2 \ln f_X(X|p,\theta)}{\partial p \partial p} & \frac{\partial^2 \ln f_X(X|p,\theta)}{\partial p \partial \theta} \\ \frac{\partial^2 \ln f_X(X|p,\theta)}{\partial \theta \partial p} & \frac{\partial^2 \ln f_X(X|p,\theta)}{\partial \theta \partial \theta} \end{pmatrix},$$

where the expectation is taken elementwise (i.e., on each element of the 2×2 matrix). Calculate $\mathbb{I}^{-1}(p,\theta)$, the inverse of $\mathbb{I}(p,\theta)$. This matrix satisfies

$$\sqrt{n}\left[\begin{pmatrix} D/n \\ K/n \end{pmatrix} - \begin{pmatrix} p \\ \theta \end{pmatrix}\right] \xrightarrow{d} \text{mvn}_2(\mathbf{0}, \mathbb{I}^{-1}(p,\theta)),$$

where $\mathbf{0} = (0,0)'$.

5. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \begin{cases} \theta(1-x)^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$.

(a) Find a sufficient statistic $T = T(\mathbf{X})$ and show that it has monotone likelihood ratio.

(b) Derive the uniformly most powerful (UMP) level α test for

$$\begin{gathered} H_0 : \theta \geq \theta_0 \\ \text{versus} \\ H_1 : \theta < \theta_0. \end{gathered}$$

You must give an explicit expression for the rejection region R for a test of size α. The rejection region must be simplified as much as possible, and all critical values must be precisely identified.

(c) Express the power function $\beta(\theta)$ in terms of the cumulative distribution function of a well known distribution.