From Casella and Berger, do the following problems from Chapter 8:

Homework 7: 3, 5, 6, and 41.
Homework 8: 13, 15, 17, 19, and 20.
Homework 9: 28, 31, 33, and 50.

These are extra problems that I have given on past exams (in STAT 713 or in related courses). You do not have to turn these in.

8.1. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\sigma) = \frac{1}{2\sigma} e^{-|x|/\sigma},$$

for $x \in \mathbb{R}$ and $\sigma > 0$.
(a) Derive a size α likelihood ratio test (LRT) of $H_0 : \sigma = 1$ versus $H_1 : \sigma \neq 1$.
(b) Derive the power function $\beta(\sigma)$ of the LRT.

8.2. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \frac{2x}{\theta} e^{-x^2/\theta} I(x > 0).$$

where $\theta > 0$. Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where θ_0 is known.
(a) Derive a size α likelihood ratio test (LRT).
(b) Derive the power function $\beta(\theta)$ of the LRT.

- Graph your power function when $n = 10$, $\theta_0 = 2$, and $\alpha = 0.05$.
- Fix $\theta_0 = 2$ and $\alpha = 0.05$. Find the smallest sample size n to guarantee $\beta(3) = 0.90$.

(c) Now consider putting an inverse gamma prior distribution on θ, namely,

$$\theta \sim \pi(\theta) = \frac{1}{\Gamma(a) b^a} \frac{1}{\theta^{a+1}} e^{-1/b\theta} I(\theta > 0),$$

where a and b are known. Show how to carry out the Bayesian test.
(d) Is the LRT you derived in part (a) a uniformly most powerful test?

8.3. Suppose that $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where both parameters are unknown. Derive the likelihood ratio test (LRT) of $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_0 : \sigma^2 > \sigma_0^2$.
(a) Argue that a LRT will reject H_0 when

$$W(\mathbf{X}) = \frac{(n-1)S^2}{\sigma_0^2}$$

is large and find the critical value to confer a size α test.
(b) Derive the power function of the LRT.

8.4. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \theta e^{-\theta x} I(x > 0),$$

where $\theta > 0$.

(a) Derive the size α likelihood ratio test (LRT) for $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$. Derive the power function of the LRT.
(b) Suppose that $n = 10$. Derive the most powerful (MP) level $\alpha = 0.10$ test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$. Calculate the power of your test.

8.5. Suppose that $X_1, X_2, ..., X_n$ are iid from

$$f_X(x|\theta) = a\theta^{-a}x^{a-1}I(0 < x < \theta),$$

where $a \geq 1$ is a known constant and $\theta > 0$ is an unknown parameter.
(a) Show that the likelihood ratio rejection region for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ can be written in terms of $X_{(n)}$, the maximum order statistic.
(b) Derive the power function of the test in part (a).
(c) Derive the most powerful (MP) level α test of $H_0 : \theta = 5$ versus $H_1 : \theta = 3$. Does your MP decision rule depend on the value of θ in H_1? What does this suggest?

8.6. Suppose that $X_1, X_2, ..., X_n$ are independent random variables (not iid) with densities

$$f_{X_i}(x|\theta_i) = \frac{\theta_i}{x^2}e^{-\theta_i/x}I(x > 0),$$

where $\theta_i > 0$, for $i = 1, 2, ..., n$.
(a) Derive the form of the likelihood ratio test (LRT) statistic for testing

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_n$$
versus
$$H_1 : \text{not } H_0.$$

You do not have to find the distribution of the likelihood ratio test (LRT) statistic under H_0. Just find the form of the statistic.
(b) From your result in part (a), deduce that

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} \geq \frac{n}{\sum_{i=1}^{n}(1/x_i)},$$

for any positive $x_1, x_2, ..., x_n$.
(c) Suppose H_0 is true so that the common distribution of $X_1, X_2, ..., X_n$, now viewed as being conditional on θ, is described by

$$f_X(x|\theta) = \frac{\theta}{x^2}e^{-\theta/x}I(x > 0),$$

where $\theta > 0$.

- Identify a conjugate prior for θ.
- Specify any hyperparameters in your prior (pick values for fun if you want). Show how to carry out the (Bayesian) test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$.

8.7. Suppose $X \sim f_X(x|\theta) = (\theta x + 1 - \frac{\theta}{2})I(0 < x < 1)$, where $0 \leq \theta < 2$. Find the UMP level $\alpha = 0.02$ test of $H_0 : \theta \geq 1$ versus $H_1 : \theta < 1$ on the basis of observing X. Determine the numerical critical value for the test.

8.8. On the basis of observing X, find the most powerful size $\alpha = 0.05$ test of $H_0 : X \sim f_0(x)$ versus $H_1 : X \sim f_1(x)$, where $f_0(x)$ is the standard normal pdf and

$$f_1(x) = \frac{1}{2}e^{-|x|}I(x \in \mathbb{R}),$$

that is $f_1(x)$ is the standard LaPlace pdf. Can you redo this problem when you have observed X_1, X_2 iid?

8.9. Let X_1 and X_2 be independent Poisson random variables with mean $\lambda > 0$. Based on these $n = 2$ observations, consider testing $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$.
(a) Let $\phi_1 = \phi_1(X_1, X_2)$ denote the test that rejects H_0 if and only if $X_1 \geq 2$. That is,

$$\phi_1(X_1, X_2) = \begin{cases} 1, & X_1 \geq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find the size of ϕ_1.
(b) Let $\phi_2(x_1, x_2) = E[\phi_1(X_1, X_2)|X_1 + X_2 = x_1 + x_2]$. Derive a simple formula for $\phi_2(x_1, x_2)$.
(c) Consider the test identified by $\phi_2 = \phi_2(X_1, X_2)$. Compare the power functions of ϕ_1 and ϕ_2 and graph them.

8.10. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \frac{1}{\theta}x^{(1-\theta)/\theta}I(0 < x < 1),$$

where $\theta > 0$.
(a) Derive the uniformly most powerful (UMP) level α test for

$$\begin{array}{c} H_0 : \theta \leq \theta_0 \\ \text{versus} \\ H_1 : \theta > \theta_0. \end{array}$$

You should be able to write your rejection region in terms of quantiles from a χ^2 distribution.
(b) Derive an expression for $\beta(\theta)$, the power function of the test in part (a). Graph your power function when $n = 20$, $\theta_0 = 2$, and $\alpha = 0.05$.

8.11. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from the distribution with density

$$f_X(x|\theta) = \frac{\theta}{x^2}I(x \geq \theta),$$

where $\theta > 0$.
(a) Find the maximum likelihood estimator (MLE) of θ.
(b) Give the form of the likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.
(c) Show that there is an appropriate statistic $T = T(\mathbf{X})$ that has monotone likelihood ratio.
(d) Derive the uniformly most powerful (UMP) level α test for

$$H_0 : \theta \leq \theta_0$$
$$\text{versus}$$
$$H_1 : \theta > \theta_0.$$

You must give an explicit expression for the rejection region R for a test of size α. The rejection region must be simplified as much as possible, and all critical values must be precisely identified.

8.12. Suppose that $X_1 \sim \text{beta}(1, \theta_1)$, $X_2 \sim \text{beta}(1, \theta_2)$, and X_1 and X_2 are independent. On the basis of observing X_1 and X_2, consider testing

$$H_0 : \theta_1 - \theta_2 = 0$$
$$\text{versus}$$
$$H_1 : \theta_1 - \theta_2 \neq 0.$$

(a) Find the likelihood ratio test (LRT) statistic $\lambda(x_1, x_2)$ to test H_0 versus H_1.
(b) Show that the LRT can be based on the statistic

$$T \equiv T(X_1, X_2) = \frac{\ln(1 - X_1)}{\ln(1 - X_1) + \ln(1 - X_2)}.$$

(c) Find the distribution of T when H_0 is true. Using this distribution, show how to perform the test when $\alpha = 0.10$.

8.13. Suppose that $X_1, X_2, ..., X_n$ is an iid sample of $\mathcal{N}(\mu_0, \sigma^2)$ observations, where μ_0 is known and $\sigma^2 > 0$ is unknown. Consider testing

$$H_0 : \sigma^2 \leq \sigma_0^2$$
$$\text{versus}$$
$$H_1 : \sigma^2 > \sigma_0^2,$$

where σ_0^2 is known.
(a) Derive a size α likelihood ratio test of H_0 versus H_1. Your rejection region should be written in terms of a sufficient statistic T and a quantile from a "named" distribution.
(b) Derive an expression for the power function of your test in part (a).
(c) Now consider putting an inverse gamma prior distribution on σ^2, namely,

$$\sigma^2 \sim \pi(\sigma^2) = \frac{1}{\Gamma(a)b^a} \frac{1}{(\sigma^2)^{a+1}} e^{-1/b\sigma^2} I(\sigma^2 > 0),$$

where $a > 0$ and $b > 0$ are known. Show how to carry out the Bayesian test of H_0 versus H_1.

14. Suppose that $X \sim \mathcal{U}(-\theta, \theta)$, where $\theta > 0$, and that we would like to test $H_0 : \theta = 1$ versus $H_1 : \theta > 1$, based on the value of X using the rejection region $R = \{x : |x| > k\}$.
(a) Find the value of k that provides a size $\alpha = 0.05$ test.
(b) Using your rejection region from part (a), graph the power function $\beta(\theta)$. What is $\beta(1.5)$?