GROUND RULES:

- This exam contains 6 questions. The questions are of equal weight.
- Print your name at the top of this page in the upper right hand corner.
- This exam is closed book and closed notes.
- Show all of your work and explain all of your reasoning! **Translation:** No work, no credit. Insufficient explanation, no credit. If you are unsure about whether or not you should explain a result or step in your derivation/proof, then this means you probably should explain it.
- Do not talk with anyone else about this exam. You must work by yourself. No communication of any type with others.

1. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & \text{otherwise,} \end{cases}$$

where $-\infty < \theta < \infty$ is unknown.

(a) The method of moments estimator of θ is $\widehat{\theta} = \overline{X} - 1$. Use the Central Limit Theorem and Delta Method to derive the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta)$.

(b) The maximum likelihood estimator of θ is $X_{(1)}$, the minimum order statistic. Find a sequence of constants a_n such that $a_n(X_{(1)} - \theta)$ converges in distribution (a non-degenerate one) and state the asymptotic distribution.

(c) Use the asymptotic distributions to approximate $\text{var}(\overline{X} - 1)$ and $\text{var}(X_{(1)})$ for large n. Which one is smaller?

2. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from a $\mathcal{N}(0, \sigma^2)$ distribution, where $\sigma^2 > 0$.

(a) Show that $U = U(\mathbf{X}) = \sum_{i=1}^n X_i^2$ is a sufficient statistic.

(b) Show directly that $T = T(\mathbf{X}) = \sum_{i=1}^n X_i$ is not sufficient by showing that $f_{\mathbf{X}|T}(\mathbf{x}|t)$ is not free of σ^2.

(c) Define $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$. Show that

$$\sqrt{n}(\widehat{\sigma}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}(0, 2\sigma^4).$$

(d) Use the result in part (c), along with Slutsky's Theorem, to show that

$$Z_n = \sqrt{\frac{n}{2}}\left(\frac{\widehat{\sigma}^2 - \sigma^2}{\widehat{\sigma}^2}\right) \xrightarrow{d} \mathcal{N}(0, 1).$$

3. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\alpha,\beta) = \alpha\beta^{\alpha}\left(\frac{1}{x+\beta}\right)^{\alpha+1} I(x>0),$$

where $\alpha > 0$ and $\beta > 0$.

(a) Show that $(X_{(1)}, X_{(2)}, ..., X_{(n)})$ is a sufficient statistic for the two-parameter family $\{f_X(x|\alpha,\beta) : \alpha > 0, \beta > 0\}$ and that no further reduction is possible.

(b) Consider the one-parameter subfamily $\{f_X(x|\alpha) : \alpha > 0\}$ arising when $\beta = \beta_0$ is known. Find a one-dimensional sufficient statistic for this subfamily.

(c) Is your sufficient statistic in part (b) complete for the subfamily $\{f_X(x|\alpha) : \alpha > 0\}$? Explain.

4. The discrete random vector (X, Y) has a joint probability mass function $f_{X,Y}(x, y)$ described in the following table:

	$y = 1$	$y = 2$
$x = 1$	$f_{X,Y}(1,1) = \frac{1}{4} - \theta$	$f_{X,Y}(1,2) = \frac{1}{4} + \theta$
$x = 2$	$f_{X,Y}(2,1) = \frac{1}{4} + \theta$	$f_{X,Y}(2,2) = \frac{1}{4} - \theta$

or, written more succinctly,

	$y = 1$	$y = 2$
$x = 1$	$\frac{1}{4} - \theta$	$\frac{1}{4} + \theta$
$x = 2$	$\frac{1}{4} + \theta$	$\frac{1}{4} - \theta$

Note that the support of (X, Y) is $\mathcal{X} = \{(1,1), (1,2), (2,1), (2,2)\}$. The parameter space is $\Theta = \{\theta : -\frac{1}{4} < \theta < \frac{1}{4}\}$. We observe a single (X, Y).

(a) Show that both X and Y are ancillary statistics.

(b) Show that $U = X + Y$ is not ancillary.

(c) Show that $V = |X - Y|$ is a sufficient statistic for θ.

(d) Show that V is a complete statistic by using the definition; that is, show that

$$E_\theta[g(V)] = 0 \ \ \forall \theta \in \Theta \implies P_\theta(g(V) = 0) = 1 \ \ \forall \theta \in \Theta.$$

5. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \begin{cases} \theta^2 x e^{-\theta x}, & x > 0 \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta > 0$.

(a) Show that $\{f_X(x|\theta) : \theta > 0\}$ is a scale family with scale parameter $1/\theta$. What is the standard density?

(b) Let $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ denote the sample mean. Show that $\overline{X}$ is a complete and sufficient statistic.

(c) Let $X_{(n)}$ denote the maximum order statistic. Show that

$$E(X_{(n)}|\overline{X}) = \frac{\overline{X} E(X_{(n)})}{E(\overline{X})}.$$

6. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from a lognormal distribution; i.e., the population probability density function is

$$f_X(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\ln x - \mu)^2/2\sigma^2} I(x > 0),$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

(a) Derive $E(X)$ and $E(X^2)$. Then find the method of moments estimators of μ and σ^2. *Hint:* Recall that if $U \sim \mathcal{N}(\mu, \sigma^2)$, then $X = e^U$ is lognormal.

(b) Find the maximum likelihood estimators of μ and σ^2. Don't worry about verifying 2nd order conditions.