GROUND RULES:

- This exam contains 6 questions. The questions are of equal weight.
- Print your name at the top of this page in the upper right hand corner.
- This exam is closed book and closed notes.
- Show all of your work and explain all of your reasoning! **Translation:** No work, no credit. Insufficient explanation, no credit. If you are unsure about whether or not you should explain a result or step in your derivation/proof, then this means you probably should explain it.
- Do not talk with anyone else about this exam. You must work by yourself. No communication of any type with others.

1. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & \text{otherwise,} \end{cases}$$

where $-\infty < \theta < \infty$.

(a) Find the method of moments estimator (MOM) of θ.

(b) Find the maximum likelihood estimator (MLE) of θ.

(c) Find a $1 - \alpha$ confidence set for θ by making use of a pivotal quantity which depends on a sufficient statistic.

2. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\mu) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}, & -\infty < x < \infty \\ 0, & \text{otherwise,} \end{cases}$$

where $-\infty < \mu < \infty$.

(a) Show that the uniformly minimum variance unbiased estimator (UMVUE) of μ^2 is

$$W \equiv W(\mathbf{X}) = \overline{X}^2 - \frac{1}{n}.$$

(b) Calculate the Cramer-Rao Lower Bound (CRLB) on the variance of all unbiased estimators of μ^2.

(c) It follows that $\text{var}_\mu(W)$ is strictly larger than the CRLB in part (b); you do not have to show this. Is this a contradiction? Explain.

(d) Calculate $\text{cov}_\mu(W, X_1 - \overline{X})$.

3. Suppose $X_1, X_2, ..., X_n$ are iid Bernoulli random variables with mean $\tau(\theta) = 1/(1+e^\theta)$, where $-\infty < \theta < \infty$.

(a) Show that the likelihood function of θ, where nonzero, can be written as

$$L(\theta|\mathbf{x}) = \left(\frac{e^\theta}{1+e^\theta}\right)^n e^{-\theta \sum_{i=1}^n x_i}.$$

(b) Find $\widehat{\theta}$, the maximum likelihood estimator (MLE) of θ by maximizing $L(\theta|\mathbf{x})$ directly. Make sure you address the cases where all of the x_i's are zero or one.

(c) Show that you get the same estimator of θ when you first maximize $L(p|\mathbf{x})$, the likelihood function of $p = E(X_1)$, and then calculate $\widehat{\theta} = \tau^{-1}(\widehat{p})$, where $\widehat{p}$ is the maximizer of $L(p|\mathbf{x})$. What property of MLEs does this illustrate?

(d) In Chapter 5, we showed that $\sqrt{n}(\widehat{p} - p) \xrightarrow{d} \mathcal{N}(0, p(1-p))$, as $n \to \infty$. Use the Delta Method to get the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta)$.

4. Suppose X_1 and X_2 are iid from

$$f_X(x|\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta > 0$, and consider testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$. We have two tests:

$$\begin{aligned} \phi_1 = \phi_1(x_1, x_2) &= I(x_1 > 0.9) \\ \phi_2 = \phi_2(x_1, x_2) &= I(x_1 x_2 > c), \end{aligned}$$

where $0 < c < 1$.

(a) Show that the power functions of the two tests are $\beta_1(\theta) = 1 - (0.9)^\theta$ and $\beta_2(\theta) = 1 + c^\theta(\theta \ln c - 1)$, respectively.

(b) Calculate the size of the ϕ_1 test. Then, find the value of c that gives the same size for the ϕ_2 test.

(c) Is ϕ_2 a most powerful test of $H_0' : \theta = 1$ versus $H_1' : \theta = 2$? Explain.

5. Suppose $X_1, X_2, ..., X_m$ are iid exponential with mean β_1. Suppose $Y_1, Y_2, ..., Y_n$ are iid exponential with mean β_2. Suppose the samples are independent.

(a) Derive the likelihood ratio test (LRT) statistic $\lambda(\mathbf{x}, \mathbf{y})$ for testing

$$\begin{gathered} H_0 : \beta_1 = \beta_2 \\ \text{versus} \\ H_1 : \beta_1 \neq \beta_2, \end{gathered}$$

and show that it is a function of $t_1 = t_1(\mathbf{x}) = \sum_{i=1}^m x_i$ and $t_2 = t_2(\mathbf{y}) = \sum_{j=1}^n y_j$.

(b) Show how you could perform a size α test in part (a) using the F distribution.

6. Suppose $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \begin{cases} \theta \left(\dfrac{1}{x+1} \right)^{\theta+1}, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$.

(a) Find a sufficient statistic $T = T(\mathbf{X})$ and show that it has monotone likelihood ratio.

(b) Derive the uniformly most powerful (UMP) level α test for

$$\begin{gathered} H_0 : \theta \leq \theta_0 \\ \text{versus} \\ H_1 : \theta > \theta_0. \end{gathered}$$

You must give an explicit expression for the rejection region R for a test of size α. The rejection region must be simplified as much as possible, and all critical values must be precisely identified.

(c) Express the power function $\beta(\theta)$ in terms of the cumulative distribution function of a well known distribution.