GROUND RULES:

- This exam contains 7 questions. Each question is worth 10 points. The maximum number of points on this exam is 70.
- Print your name at the top of this page in the upper right hand corner.
- This is a closed-book and closed-notes exam. You may not use a calculator.
- Show all of your work. Explain all of your reasoning.
- Any discussion or otherwise inappropriate communication between examinees, as well as the appearance of any unnecessary material, will be dealt with severely.
- You have 80 minutes to complete this exam. GOOD LUCK!

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. Define

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Suppose that $\mathbf{M}$ is the perpendicular projection matrix onto $\mathcal{C}(\mathbf{A})$. Find $r(\mathbf{M})$ and $tr(\mathbf{M})$. Explain clearly why your answers are correct.

2. Consider the linear model

$$Y_{ij} = \mu + \alpha_i + \alpha_j + \epsilon_{ij},$$

for $i = 1, 2$ and $j = 1, 2$, where μ, α_1, and α_2 are fixed and unknown parameters.
(a) Write this model using the matrix notation $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$.
(b) Are the columns of $\mathbf{X}$ linearly independent? If they are, prove it. If they are not, demonstrate where the linear dependencies are.

3. Let $\mathbf{P_X}$ and $\mathbf{P_W}$ be perpendicular projection matrices with $\mathcal{C}(\mathbf{P_W}) \subset \mathcal{C}(\mathbf{P_X})$. Show that $\mathbf{P_X} - \mathbf{P_W}$ is a perpendicular projection matrix.

4. Define the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) Find $\mathcal{C}(\mathbf{A})$ and a basis for this space.
(b) Find $\mathcal{N}(\mathbf{A}')$ and a basis for this space.

5. Consider the ANCOVA model

$$Y_{ij} = \mu + \alpha_i + \beta x_{ij} + \epsilon_{ij},$$

where $E(\epsilon_{ijk}) = 0$, for $i = 1, 2$ and $j = 1, 2, 3$. Assume that $\sum_{j=1}^{3}(x_{ij} - \overline{x}_{i+})^2 > 0$, for $i = 1, 2$.
(a) Write this model in $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ form.
(b) Find necessary and sufficient conditions for $\boldsymbol{\lambda}'\boldsymbol{\beta}$ to be estimable.
(c) A maximal set of linearly independent estimable functions contains r functions. What is r in this problem?

6. Consider the general linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $\text{cov}(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$. Suppose that $\mathbf{X}$ is $n \times p$ with rank $r < p$. Let $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$ denote a least squares estimator of $\boldsymbol{\beta}$.
(a) Find $E(\widehat{\boldsymbol{\beta}})$ and $\text{cov}(\widehat{\boldsymbol{\beta}})$.
(b) Do your results in part (a) change when $r = p$? Verify any claims you make.

7. Consider the general linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $E(\boldsymbol{\epsilon}) = \mathbf{0}$. Let $\widehat{\mathbf{e}}$ denote the vector of residuals obtained from the least squares fit. Prove that $\widehat{\boldsymbol{\beta}}$ is a least squares estimate of $\boldsymbol{\beta}$ if and only if $\widehat{\mathbf{e}} \perp \mathcal{C}(\mathbf{X})$.