1. Define the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) Find two generalized inverses of $\mathbf{A}$.
(b) Find a matrix which projects onto $\mathcal{C}(\mathbf{A})$.
(c) Find a matrix which projects onto $\mathcal{C}(\mathbf{A})^{\perp}$, the orthogonal complement of $\mathcal{C}(\mathbf{A})$.

2. Show that if $\mathbf{A}^{-}$ is a generalized inverse of $\mathbf{A}$, then so is

$$\mathbf{G} = \mathbf{A}^{-}\mathbf{A}\mathbf{A}^{-} + (\mathbf{I} - \mathbf{A}^{-}\mathbf{A})\mathbf{B}_1 + \mathbf{B}_2(\mathbf{I} - \mathbf{A}\mathbf{A}^{-}),$$

for any choices of $\mathbf{B}_1$ and $\mathbf{B}_2$ with conformable dimensions.

3. Let $\mathbf{A}_{n\times p}$, $\mathbf{b}_{p\times 1}$, $\mathbf{c}_{n\times 1}$, and suppose that the equations $\mathbf{Ab} = \mathbf{c}$ are consistent. Let $\mathbf{x}_{n\times 1}$, $\mathbf{u}_{p\times 1}$, and $\mathbf{X}_{p\times n}$. Let $\mathbf{A}_1^{-}$ and $\mathbf{A}_2^{-}$ be two generalized inverses of $\mathbf{A}$. Let $\mathbf{I}$ denote the $n \times n$ identity matrix.
(a) Let $\mathbf{b}^*$ be a solution to $\mathbf{Ab} = \mathbf{c}$. Show that $\mathbf{b}^* + \mathbf{u}\mathbf{c}'\{(\mathbf{A}_1^{-})'\mathbf{A}' - \mathbf{I}\}\mathbf{x}$ is also a solution.
(b) Show that $\mathbf{A}_1^{-} + \mathbf{X}(\mathbf{A}\mathbf{A}_2^{-} - \mathbf{I})$ is a generalized inverse of $\mathbf{A}$.

4. Suppose the system $\mathbf{Ax} = \mathbf{c}$ is consistent and that $\mathbf{G}$ is a generalized inverse of $\mathbf{A}$.
(a) What is a particular solution to the system? the general solution?
(b) If $\mathbf{A}$ is symmetric, prove that $\frac{1}{2}(\mathbf{G} + \mathbf{G}')$ is a generalized inverse of $\mathbf{A}$.
(c) Prove that the generalized inverse in (b) is symmetric. This shows that there does exist a generalized inverse of $\mathbf{A}$, $\mathbf{A}$ symmetric, that is symmetric itself.

5. Suppose that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} + \mathbf{B}$ are all idempotent. Prove that $\mathbf{AB} = \mathbf{0}$ and $\mathbf{BA} = \mathbf{0}$.

6. Let $\mathbf{P}$ be an $n \times n$ orthogonal matrix and let $\mathbf{A}$ be an $n \times n$ symmetric and idempotent matrix. Define $\mathbf{D} = \mathbf{P}'\mathbf{A}\mathbf{P}$. Show that $\mathbf{D}$ is a perpendicular projection matrix.

7. Consider the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with

$$\mathbf{Y} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Note that $r(\mathbf{X}) = 3$. Find $\widehat{\boldsymbol{\beta}}_1$ and $\widehat{\boldsymbol{\beta}}_2$, two different solutions to the normal equations $\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{Y}$. With your solutions, show that $\mathbf{X}\widehat{\boldsymbol{\beta}}_1 = \mathbf{X}\widehat{\boldsymbol{\beta}}_2 \in \mathcal{C}(\mathbf{X})$. Also show that $\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_1 = \mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_2 \in \mathcal{N}(\mathbf{X}')$.

8. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be perpendicular projection matrices on $\mathcal{R}^n$. Prove that $\mathbf{M}_1 + \mathbf{M}_2$ is the perpendicular projection matrix onto $\mathcal{C}(\mathbf{M}_1, \mathbf{M}_2)$ if and only if $\mathcal{C}(\mathbf{M}_1) \perp \mathcal{C}(\mathbf{M}_2)$.

9. Let $\mathbf{M}$ be the perpendicular projection matrix onto $\mathcal{C}(\mathbf{X})$. Suppose that $\mathbf{a} \in \mathcal{C}(\mathbf{X})$. Show that $(\mathbf{M} - \mathbf{aa}')'(\mathbf{M} - \mathbf{aa}') = \mathbf{M} + (\mathbf{a}'\mathbf{a} - 2)\mathbf{aa}'$.

10. Suppose that $\mathbf{M}_1$ and $\mathbf{M}_2$ are symmetric, that $\mathcal{C}(\mathbf{M}_1) \perp \mathcal{C}(\mathbf{M}_2)$, and that $\mathbf{M}_1 + \mathbf{M}_2$ is the perpendicular projection matrix. Prove that $\mathbf{M}_1$ and $\mathbf{M}_2$ are also perpendicular projection matrices.

11. Let $\mathbf{M}$ and $\mathbf{M}_0$ be perpendicular projection matrices with $\mathcal{C}(\mathbf{M}_0) \subset \mathcal{C}(\mathbf{M})$. Show that $\mathbf{M} - \mathbf{M}_0$ is a perpendicular projection matrix.

DEFINITION: Let $\mathcal{V}$ denote an arbitrary vector space and let $\mathcal{S}$ denote a subspace of $\mathcal{V}$. Define

$$\mathcal{S}_{\mathcal{V}}^{\perp} = \{\mathbf{y} \in \mathcal{V} : \mathbf{y} \perp \mathcal{S}\}.$$

The subspace $\mathcal{S}_{\mathcal{V}}^{\perp}$ is called the **orthogonal complement of $\mathcal{S}$ with respect to $\mathcal{V}$**. If $\mathcal{V} = \mathcal{R}^n$, then $\mathcal{S}_{\mathcal{V}}^{\perp} \equiv \mathcal{S}^{\perp}$; in this situation, we call $\mathcal{S}^{\perp}$ and $\mathcal{S}$ simply "orthogonal complements" because it is understood that the larger vector space is $\mathcal{R}^n$. However, there is nothing to prevent $\mathcal{V}$ from being a subspace of $\mathcal{R}^n$.

12. Let $\mathbf{M}$ and $\mathbf{M}_0$ be perpendicular projection matrices with $\mathcal{C}(\mathbf{M}_0) \subset \mathcal{C}(\mathbf{M})$. Show that $\mathcal{C}(\mathbf{M} - \mathbf{M}_0) = \mathcal{C}(\mathbf{M}_0)_{\mathcal{C}(\mathbf{M})}^{\perp}$, the orthogonal complement of $\mathcal{C}(\mathbf{M}_0)$ with respect to $\mathcal{C}(\mathbf{M})$.