1. Let $\mathbf{Y} = (Y_1, Y_2, Y_3)' \sim \mathcal{N}_3(\boldsymbol{\mu}, \mathbf{V})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Define the statistics

$$\begin{aligned} T_1 &= 3Y_1 + Y_2 - 2Y_3 \\ T_2 &= Y_1 - 5Y_2 + Y_3. \end{aligned}$$

(a) Find the distribution of $\mathbf{T} = (T_1, T_2)'$.
(b) Differentiate the mgf of $\mathbf{T}$ to find $E(\mathbf{T})$ and $\text{cov}(\mathbf{T})$. These should match your answers in part (a).

2. Consider the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{X}$ is $n \times p$ matrix with (full) rank $r = p$ and $\boldsymbol{\epsilon} \sim \mathcal{N}_n(\mathbf{0}, \sigma^2\mathbf{I})$.
(a) Find the distribution of $\widehat{\boldsymbol{\beta}}$.
(b) Find the distribution of $\widehat{\theta} = \mathbf{a}'\widehat{\boldsymbol{\beta}}$, for a fixed conformable $\mathbf{a}$.
(c) Find the distribution of $\widehat{\mathbf{Y}}$ (the vector of fitted values).
(d) Find the distribution of $\widehat{\mathbf{e}}$ (the vector of residuals).

3. Suppose that $\phi(\cdot)$ is the standard normal density function.
(a) For $b < 1/2$, compute

$$I(a, b) = \int_{\mathcal{R}} e^{az+bz^2} \phi(z) dz.$$

(b) ($\uparrow$) Suppose further that Y_1 and Y_2 are independent random variables with moment generating functions $M_{Y_1}(t) = I(t, 0)$ and $M_{Y_2}(t) = I(0, t)$. Find the constant k so that the statistic $U = k(Y_1^2 + Y_2)$ follows an exponential distribution with mean 2.

4. Let U and V have the joint density

$$f_{U,V}(u, v) = \begin{cases} \pi^{-1} e^{-\frac{1}{2}(u^2+v^2)}, & uv \geq 0 \\ 0, & uv < 0. \end{cases}$$

(a) Show that U and V are both marginally standard normal.
(b) Is the covariance matrix of $(U, V)'$ singular or nonsingular?
(c) Does $(U, V)'$ have a bivariate normal distribution? What is the main point?

5. Suppose that U has a noncentral χ^2 with n degrees of freedom and noncentrality parameter λ. Determine conditions under which $E(U^{-j}) < \infty$, $j = 1, 2, ...,$ and give expressions for these (inverse) moments when they are finite.

6. Suppose that $\mathbf{Y} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{V})$, $r(\mathbf{V}) = p$, and let $\mathbf{A}$ be a symmetric matrix defining the quadratic form $\mathbf{Y}'\mathbf{A}\mathbf{Y}$. Let $\lambda_1, \lambda_2, ..., \lambda_p$ denote the eigenvalues of $\mathbf{V}^{1/2}\mathbf{A}\mathbf{V}^{1/2}$.
(a) Give the conditions, in terms of $\lambda_1, \lambda_2, ..., \lambda_p$, under which $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ has a χ^2 distribution.
(b) When $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is not χ^2, it is sometimes approximated by the distribution of a constant multiple of a χ^2 random variable. The constant and the degrees of freedom are chosen to match the first two moments of $\mathbf{Y}'\mathbf{A}\mathbf{Y}$. Let U denote a random variable having a (central) χ^2 distribution with r degrees of freedom. Determine the constants c and r in terms of $\lambda_1, \lambda_2, ..., \lambda_p$ so that $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ and cU have the same mean and variance.

7. Suppose that $Z_1, Z_2, ..., Z_n$ are iid standard normal random variables.
(a) Derive the joint distribution of $\overline{Z}, Z_1 - \overline{Z}, Z_2 - \overline{Z}, ..., Z_n - \overline{Z}$.
(b) ($\uparrow$) Deduce that $\overline{Z}$ and $\sum_{i=1}^n (Z_i - \overline{Z})^2$ are independent.
(c) Let $\mathbf{Y}$ have an n-variate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$, where $\mathrm{var}(Y_i) = \sigma^2$, for all i, and $\mathrm{cov}(Y_i, Y_j) = \sigma^2(1-\rho)$, for $i \neq j$, where $0 < \rho < 1$. Prove that $\overline{Y}$ and $\sum_{i=1}^n (Y_i - \overline{Y})^2$ are independent. This is a generalization of part (b).

8. Suppose that $\mathbf{X}$ is a random vector with moment generating function $M_{\mathbf{X}}(\mathbf{t})$. Let $\dot{M}_{\mathbf{X}}(\mathbf{t}) = (\partial/\partial\mathbf{t})M_{\mathbf{X}}(\mathbf{t})$ and $\ddot{M}_{\mathbf{X}}(\mathbf{t}) = (\partial/\partial\mathbf{t}')\dot{M}_{\mathbf{X}}(\mathbf{t})$. Prove each of the following:

$$
\begin{aligned}
E(\mathbf{X}) &= \dot{M}_{\mathbf{X}}(\mathbf{t})|_{\mathbf{t}=\mathbf{0}} \\
E(\mathbf{X}\mathbf{X}') &= \ddot{M}_{\mathbf{X}}(\mathbf{t})|_{\mathbf{t}=\mathbf{0}} \\
E(\mathbf{X}'\mathbf{A}\mathbf{X}) &= tr\left\{\mathbf{A}\ddot{M}_{\mathbf{X}}(\mathbf{t})|_{\mathbf{t}=\mathbf{0}}\right\}.
\end{aligned}
$$

9. *Calibration.* Consider the simple linear regression model

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

for $i = 1, 2, ..., n$, where $\epsilon_i \sim$ iid $\mathcal{N}(0, \sigma^2)$. Suppose that a new value of Y is observed, say, Y_0 (independent of $Y_1, Y_2, ..., Y_n$) and that we wish to estimate the corresponding value of x_0.
(a) Regard x_0 as a parameter. Based on the observed data (x_i, Y_i), $i = 1, 2, ..., n$, and the new value Y_0, find the maximum likelihood estimators of β_0, β_1, x_0, and σ^2.
(b) Derive a $100(1-\alpha)$ percent confidence interval for x_0. When does such an interval exist? *Hint*: Consider the random variable $V = (Y_0 - \widehat{Y}_0)^2$, where $\widehat{Y}_0 = \widehat{\beta}_0 + \widehat{\beta}_1 x_0$. Find a function of V that has an $F(1, n-2)$ distribution.

10. Suppose that Y_1 and Y_2 are iid $\mathcal{N}(0,1)$ random variables. Show that the moment generating function of $X = Y_1 Y_2$ is

$$M_X(t) = (1-t^2)^{-1/2}$$

11. Suppose that $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$ and let $\mathbf{P_X}$ be the perpendicular projection operator onto $\mathcal{C}(\mathbf{X})$. Suppose that $r(\mathbf{X}) = r$. Define

$$\widetilde{\sigma}^2 = \mathbf{Y}'(\mathbf{I} - \mathbf{P_X})\mathbf{Y}/k,$$

and consider using $\widetilde{\sigma}^2$ as an estimator for σ^2.
(a) Show that the mean-squared error of $\widetilde{\sigma}^2$ is given by

$$\text{MSE}(\widetilde{\sigma}^2) = \sigma^4 \left[\frac{2(n-r) + (n-r-k)^2}{k^2}\right].$$

(b) Find the value of k that minimizes $\text{MSE}(\widetilde{\sigma}^2)$.

12. Assume that $\mathbf{Y} \sim \mathcal{N}_2(\boldsymbol{\mu}, \mathbf{V})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\mathbf{A} = \frac{1}{8}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

and $\mathbf{B} = (1, -2)'$.
(a) Find the distribution of $U_1 = \mathbf{Y}'\mathbf{A}\mathbf{Y}$.
(b) Find the distribution of $U_2 = \mathbf{B}'\mathbf{Y}$.
(c) Compute the correlation between U_1 and U_2.

13. Consider the two-way fixed effects (crossed) ANOVA model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij},$$

for $i = 1, 2, ..., a$ and $j = 1, 2, ..., b$, where $\epsilon_i \sim$ iid $\mathcal{N}(0, \sigma^2)$. In matrix form, $\mathbf{X}$ and $\boldsymbol{\beta}$ are

$$\mathbf{X}_{n\times p} = \begin{pmatrix} \mathbf{1}_b & \mathbf{1}_b & \mathbf{0}_b & \cdots & \mathbf{0}_b & \mathbf{I}_b \\ \mathbf{1}_b & \mathbf{0}_b & \mathbf{1}_b & \cdots & \mathbf{0}_b & \mathbf{I}_b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{1}_b & \mathbf{0}_b & \mathbf{0}_b & \cdots & \mathbf{1}_b & \mathbf{I}_b \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta}_{p\times 1} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_a \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_b \end{pmatrix},$$

where $p = a + b + 1$ and $n = ab$. Redo Example 5.6 (notes), showing how to choose the $\mathbf{A}$ matrices, determining the distributions of the relevant quadratic forms, and showing the form the associated ANOVA table with expected mean squares. Show also how to construct F tests for $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$ and $H_0 : \beta_1 = \beta_2 = \cdots = \beta_b = 0$.

14. Suppose that $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{I})$ and that $\mathbf{A}_i$ is a symmetric idempotent matrix with rank s_i, for $i = 1, 2, ..., k$. Assume that $\mathbf{A}_i\mathbf{A}_j = \mathbf{0}$, for $i \neq j$ and that $\mathbf{A}_i\boldsymbol{\mu} = \boldsymbol{\beta}_i$, for $i = 1, 2, ..., k$. Define $\mathbf{Y} = \mathbf{A}_1\mathbf{X} + \mathbf{A}_2\mathbf{X} + \cdots + \mathbf{A}_k\mathbf{X}$. Determine the distribution of $\mathbf{Y}'\mathbf{Y}$ in terms of $s_1, s_2, ..., s_k$ and $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, ..., \boldsymbol{\beta}_k$.