

## Estimability Example

An education major wanted to test the efficacy of teaching methods for the division of fractions. Two new methods along with the standard method were studied. Five teachers were trained in all methods and taught a total of twelve classes. Differences in pre- and post-test scores are recorded below:

Method	Teacher				
	1	2	3	4	5
A	10, 7	6	11		6
B	4		5	7, 8	3
C		13	16		

Our model is:

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk}, \quad i = 1, 2, 3; \quad j = 1, 2, 3, 4, 5; \quad k = 1, \dots, n_{ij},$$

where

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

The population marginal mean for Method A would be:

$$\begin{aligned} PMM(\alpha_1) &= \frac{1}{5}(\mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{15}) \\ &= \mu + \alpha_1 + \frac{1}{5}(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) + \frac{1}{5}(\gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{15}) \end{aligned}$$

Remember that this is the parameter that LSMEANS( $\alpha_1$ ) would estimate. To find out whether or not this parameter is estimable in the interaction model (of course, we know it's not since there is no entry for cell  $\mu_{14}$ ), we ask SAS for the general form of estimable functions:

```
proc glm;
classes method teacher;
model diff=method|teacher/e;
```

We obtain the following SAS output. The coefficients are parameter coefficients for the unconstrained model. Note that L9=L1-L5-L6-L7-L8, for instance, because the unconstrained parameterization is redundant and hence suppressed.

Effect	Coefficients
--------	--------------

INTERCEPT		L1
METHOD	a	L2
	b	L3
	c	L1-L2-L3
TEACHER	1	L5
	2	L6
	3	L7
	4	L8
	5	L1-L5-L6-L7-L8
METHOD*TEACHER	a 1	L10
	a 2	L11
	a 3	L12
	a 5	L2-L10-L11-L12
	b 1	L5-L10
	b 3	-L1+L2+L3+L6+L7-L11-L12
	b 4	L8
	b 5	L1-L2-L5-L6-L7-L8+L10+L11+L12
	c 2	L6-L11
c 3	L1-L2-L3-L6+L11	

In order for any linear combination of the parameters to be estimable, we need to be able to express it as:

$$\begin{aligned}
& L1\mu + L2\alpha_1 + L3\alpha_2 + (L1-L2-L3)\alpha_3 + L5\beta_1 + L6\beta_2 \\
& + L7\beta_3 + L8\beta_4 + (L1-L5-L6-L7-L8)\beta_5 + L10\gamma_{11} + L11\gamma_{12} + L12\gamma_{13} \\
& + 0 \cdot \gamma_{14} + (L2-L10-L11-L12)\gamma_{15} + (L5-L10)\gamma_{21} + 0 \cdot \gamma_{22} + (-L1+L2+L3+L6+L7-L11-L12)\gamma_{23} \\
& + L8\gamma_{24} + (L1-L2-L5-L6-L7-L8+L10+L11+L12)\gamma_{25} + 0 \cdot \gamma_{31} + (L6-L11)\gamma_{32} \\
& + (L1-L2-L3-L6+L11)\gamma_{33} + 0 \cdot \gamma_{34} + 0 \cdot \gamma_{35}
\end{aligned}$$

$PMM(\alpha_1)$  is not estimable because, for example, the coefficient of  $\gamma_{14}$  would have to be .2, but estimable functions set the coefficient to 0. To systematically check estimability, look at the coefficients for the population marginal mean, make the appropriate assignments for L1, L2, etc and then see if any inconsistencies occur. If we were checking for estimability of  $PMM(\alpha_1)$ , for example, we would need L1=1, L2=1, L3=0, L5=L6=L7=L8=.2, L10=.2, L11=L12=0. The method breaks down, though, because there is no way to match our coefficient of .2 for  $\gamma_{14}$  with its assigned coefficient of L13=0; there are many other ways to arrive at such contradictions.

Individual exercise: What is the population marginal mean for Teacher 3? Is the population marginal mean for Teacher 3 estimable? If it is, find and verify the coefficients.