1. Consider the “trapezoidal” density function:

\[ g(x) = -1.25x + 2.875, \quad 1 \leq x \leq 2. \]

- Show how to generate random deviates from this distribution using the probability integral transform.
- Show how to generate random deviates from this distribution using a mixture of a uniform distribution and a triangular distribution.
- Discuss the advantages/disadvantages of the above methods.
- Implement both methods by generating 500 deviates from \( G(\cdot) \); evaluate your results.

2. We had earlier considered several different strategies for evaluating

\[ \int_{1}^{2} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \]

Monte Carlo integration from a \( Unif(1, 2) \) distribution was more efficient than either Monte Carlo integration or importance sampling based on a \( N(0, 1) \) distribution. Consider an importance sampling approach based on the trapezoidal distribution from Problem 1 with \( h(x) = e^{-x^2/2}/\sqrt{2\pi} \), \( f(x) \) a \( Unif(1, 2) \) density and \( g(x) \) as defined above.

- Based on results from our notes and the text, explain why this approach could be expected to be efficient.
- Compute the variance of the theoretical lower bound for an importance sampling estimator, the variance of the proposed importance sampling estimator and the variance of the usual \( Unif(1, 2) \) Monte Carlo estimator. Use analytical methods when possible. Comment.
- Confirm your theoretical results with a simulation exercise. You should assume samples of size 10 and 25 and use 1000 simulations.

3. Consider optimizing the likelihood for a random sample of \( N(\mu, \sigma^2) \) random variables.

- Write the negative of the log likelihood as a function of \( \mu \) and \( \omega \), where \( \omega = \sigma^2 \). Find the gradient and Hessian for this function.
- Write a script or function in \( R \) to compute the Newton-Raphson “step” for a single iterate—do NOT compute the inverse of the Hessian to do this.
- Generate 20 standard normal random deviates; calculate the MLE of \((\mu, \omega)\). Using a start value of \((0, 1)\), calculate the first NR step for your function. Compare to the MLE and comment. Take a few more steps until you apparently have convergence (I know this goes against what I taught you in class, but compare your output to the calculated MLEs till they are equivalent to 6 significant digits). How many steps did your algorithm use?
- Use \( nlm \) in \( R \) to find the MLE’s compare the output from \( nlm \) to your results.