Section 3.5 Class Exercise

We will be deriving the sampling distribution of \( n = \{n_{ij}\} \) under an independence model for fixed row and column marginals \( \{n_{i+}\} \) and \( \{n_{+j}\} \).

Start by assuming \( n \) is distributed Multinomial\((n, \pi)\), where \( n \) is the sum of the elements of \( n \).

\[
f(n) = \frac{n! \prod_i \prod_j n_{ij}^{n_{ij}}}{\prod_i \prod_j n_{ij}!}
\]

The row marginals have the joint distribution given below:

\[
f(n_{i+}) = \frac{n! \prod_i n_{i+}^{n_{i+}}}{\prod_i n_{i+}!}
\]

And the column marginals have joint distribution

\[
f(n_{+j}) = \frac{n! \prod_j n_{+j}^{n_{+j}}}{\prod_j n_{+j}!}
\]

Irrespective of the independence assumption, conditional on the row marginals, \( n_i \) is distributed Multinomial\((n_{i+}, \frac{\pi}{n_{i+}})\), \( i = 1, \ldots, I \).

1. Confirm that under the independence assumption \((\pi_{ij} = \pi_{i+} \cdot \pi_{+j})\), we have:

   \( n_i \) distributed Multinomial \((n_{i+}, \frac{\pi}{n_{i+}})\), \( i = 1, \ldots, I \).

2. Based on the results above, confirm that \( n \) has the following product multinomial sampling distribution under independence and fixed row marginals:

   \[
f(n|n_1) = \frac{(\prod_i n_{i+}!) \left( \prod_j n_{+j}^{n_{+j}} \right)}{\prod_i \prod_j n_{ij}!}
\]

3. Now condition on the distribution of the column marginals to confirm that \( n \) given \( n_{i+} \) and \( n_{+j} \) has the following distribution under the independence model:

   \[
f(n|n_{i+}, n_{+j}) = \frac{(\prod_i n_{i+}! \left( \prod_j n_{+j}^{n_{+j}} \right))}{n! \prod_i \prod_j n_{ij}!}
\]