Section 3.5 Class Exercise

We will be deriving the sampling distribution of $\mathbf{n} = \{\mathbf{n_{ij}}\}$ under an independence model for fixed row and column marginals $\{n_{i+}\}$ and $\{n_{+j}\}$.

Start by assuming $\mathbf{n}$ is distributed Multinomial(n, $\boldsymbol{\pi}$), where n is the sum of the elements of $\mathbf{n}$.

$$f(\mathbf{n}) = \frac{n! \prod_i \prod_j \pi_{ij}^{n_{ij}}}{\prod_i \prod_j n_{ij}!}$$

The row marginals have the joint distribution given below:

$$f(\mathbf{n_{i+}}) = \frac{n! \prod_i \pi_{i+}^{n_{i+}}}{\prod_i n_{i+}!}$$

And the column marginals have joint distribution

$$f(\mathbf{n_{+j}}) = \frac{n! \prod_j \pi_{+j}^{n_{+j}}}{\prod_j n_{+j}!}$$

Irrespective of the independence assumption, conditional on the row marginals, $\mathbf{n_i}$ is distributed Multinomial(n_{i+}, $\frac{\boldsymbol{\pi}_i}{\pi_{i+}}$), $i = 1, \ldots, I$.

1. Confirm that under the independence assumption ($\pi_{ij} = \pi_{i+} \cdot \pi_{+j}$), we have:

$$\mathbf{n_i} \text{ distributed Multinomial}\left(\mathbf{n_{i+}}, \pi_{+\mathbf{1}}, \ldots, \pi_{+\mathbf{J}}\right),\ \mathbf{i} = \mathbf{1}, \ldots, \mathbf{I}.$$

2. Based on the results above, confirm that $\mathbf{n}$ has the following product multinomial sampling distribution under independence and fixed row marginals:

$$f(\mathbf{n}|\mathbf{n_i}) = \frac{\left(\prod_i n_{i+}!\right)\left(\prod_j \pi_{+j}^{n_{+j}}\right)}{\prod_i \prod_j n_{ij}!}$$

3. Now condition on the distribution of the column marginals to confirm that $\mathbf{n}$ given $\mathbf{n_{i+}}$ and $\mathbf{n_{+j}}$ has the following distribution under the independence model:

$$f(\mathbf{n}|\mathbf{n_{i+}}, \mathbf{n_{+j}}) = \frac{\left(\prod_i n_{i+}!\right)\left(\prod_j n_{+j}!\right)}{n! \prod_i \prod_j n_{ij}!}$$