STAT 515 Lec 02 Counting Rules

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Counting rules

For some statistical experiments we can compute the probabilities of events by counting the number of sample points corresponding to the event and dividing by the total number of sample points in the sample space. Specifically:

Probability rule 1: Computing probs. if all sample points equally likely

If each sample point in the sample space S of a statistical experiment is equally likely, then the probability of an event A is equal to

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}.$$

Example. Roll a die and record the roll. Let A be the event that a 1 or a 2 is rolled. Then

$$\begin{aligned} P(A) &= P(\{\text{roll a 1}\} \cup \{\text{roll a 2}\}) \\ &= P(\text{roll a 1}) + P(\text{roll a 2}) \\ &= 1/6 + 1/6 \\ &= 1/3. \end{aligned}$$

Likewise, we can simply take the number of sample points in A, which is 2, and divide it by the number of sample points in the sample space S, which is 6, to get 1/3.

Example. Roll two dice and record the rolls. Let A be the event that the sum of the rolls is equal to 8. We may write

$$A = \{(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)\},$$

so there are 5 ways to roll two dice such that the sum is 8. There are 36 possible outcomes of rolling two dice, each occurring with equal probability. Thus $P(A) = 5/36$.

Exercise. You (Y) and your friend (F) and two strangers (S_1 and S_2) decide to play 2-on-2 beach volleyball, randomly picking the teams. What is the probability that you will be on your friend's team?

Answer: Let's list every possible assignment of the four of you into two teams of two:

Team 1	Team 2
YF	S_1S_2
YS_1	FS_2
YS_2	FS_1
S_1S_2	YF
FS_2	YS_1
FS_1	YS_2

So there are 6 sample points in the sample space, and you are on your friend's team in 2 of the 6. Since each set of teams is equally likely, you have a 1/3 chance of getting on the same team as your friend.

It is often very impractical to list all the points in the sample space of a statistical experiment. In such cases we can use some counting rules.

The probabilists of old liked to draw balls from urns:

Counting rule 1: Fundamental theorem of counting

Draw one ball from each of K urns having $n_1, \dots, n_K$ balls in them. The number of ways to do this is

$$n_1 \times n_2 \times n_3 \times \cdots \times n_K.$$

Instead of thinking about drawing balls from urns, we may simply think of a series of tasks we must complete, where each task may be completed in multiple ways. The total number of ways in which we can complete the series of tasks is the product of the numbers of ways in which we can complete the different tasks.

Exercise. Suppose there are 2 sections of STAT 515 and 3 sections of BIOL 303 and that you have to take both courses this semester. How many ways can you choose your sections of the two courses?

Answer: $2 \times 3 = 6$.

Counting rule 2: Permutation

Draw r elements of N elements and arrange them in some order. The number of ways to do this is

$$N(N-1)\cdots(N-r+1) = \frac{N!}{(N-r)!}.$$

Note that $N! = N \times (N-1) \times (N-2) \cdots \times 1$.

Exercise. Suppose that in a class of 17 students, the students with the highest, second-highest, and third-highest scores on the next test get a gold, silver, and bronze sticky star, respectively. In how many ways can the three stars be given to three students?

Answer: $17 \times 16 \times 15 = 4{,}080$.

The probabilists of old also liked to put balls *into* urns:

Counting rule 3: Partition

Place N balls into K urns such that the K urns receive $n_1, \ldots, n_K$ balls, respectively, where $N = n_1 + \cdots + n_N$. The number of ways to do this is

$$\frac{N!}{n_1! n_2! \cdots n_K!}.$$

Exercise. Suppose 11 of you are driving to the beach for some sunny funtimes and you are taking a Nissan Rogue, a Honda Pilot, and a motorcycle. In how many ways can you put 4 people into the Nissan Rogue, 5 people into the Honda Pilot, and have 2 people ride the motorcycle?

Answer:

$$\begin{aligned}
\frac{11!}{4!5!2!} &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1} \\
&= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 2} \\
&= 11 \times 10 \times 9 \times 7 \\
&= 6{,}930
\end{aligned}$$

Counting rule 4: Combination

Draw n elements from N without regard to their order. The number of ways to do this is

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}.$$

Note that a combination is simply a partitioning of N "balls" into with $K = 2$ "urns", where $n_1 = n$ balls are placed into the one urn and $n_2 = N - n$ balls are placed into the other urn.

Exercise. Suppose your parents insist that you come home on at least 3 weekends during the semester, of which there are 16 total weekends. In how many ways can you select the 3 weekends during which you will visit your parents?

Answer:

$$\frac{16!}{3!(16-3)!} = \frac{16 \times 15 \times 14}{3 \times 2} = 16 \times 5 \times 7 = 80 \times 7 = 560.$$

More exercises

Exercise. Suppose you attend a football game with 3 of your friends and you all sit in the same row next to each other. In how many possible ways can you arrange yourselves?

Answer: This is a permutation. It can be done in $4 \times 3 \times 2 \times 1 = 24$ ways.

Exercise. Suppose you have 19 people and you want to split them into 2 teams of 6 and one team of 7. In how many ways can you do this?

Answer: This is a partition. It can be done in

$$\frac{19!}{6!6!7!} = 46{,}558{,}512$$

ways!

Exercise. Suppose you are playing a card game in which the full 52 card deck is used and you are dealt 5 cards. How many different hands are there?

Answer: This is a combination. The number of different hands is

$$\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2{,}598{,}960.$$

Exercise. Suppose your avatar for the coolest game since Pokémon Go can have 6 choices of pants, 4 choices of shirt, 3 choices of backpack, 4 choices of skin tone, 4 choices of hairstyle, and may wear sunglasses or not. How many different avatars can you create?

Answer: It is multiplicative. The number of possible avatars is

$$6 \times 4 \times 3 \times 4 \times 4 \times 2 = 2{,}304.$$

Computing some probabilities by counting sample points

Exercise. Suppose you are dealt a hand of 5 cards from a standard 52-card deck. What is the probability your hand has the queen of spades?

Answer: We need to count the number of 5-card hands which contain the queen of spades and divide this by the total number of 5-card hands:

$$P(\text{ Queen of spades in 5-card hand }) = \frac{\#\{\text{ 5-card hands with queen of spades }\}}{\#\{\text{ 5-card hands }\}}$$

We have

$$\#\{\text{ 5-card hands }\} = \binom{52}{5} = \texttt{choose(52,5)} = 2{,}598{,}960.$$

We can compute the number of 5-card hands containing the queen of spades by imagining pulling the queen of spades out of the deck, and then drawing any four cards from the remaining cards. So we get

$$\#\{\text{ 5-card hands with queen of spades }\} = \binom{51}{4} = \texttt{choose(51,4)} = 249{,}900.$$

The probability of being dealt a 5-card hand containing the queen of spades is therefore

$$P(\text{ Queen of spades in 5-card hand }) = \frac{249{,}900}{2{,}598{,}960} = 0.09615385.$$

Exercise. Suppose you are dealt a hand of 5 cards from a standard 52-card deck. What is the probability that your hand contains cards of only one suit?

Answer: We need to count the number of single-suited 5-card hands and divide this by the total number of hands. We can compute the number of 5-card hands which contain cards of only one suit by considering, for each suit, the number of 5-card hands containing cards of only that suit. Each suit contains 13 cards, so for each of the 4 suits, there are $\binom{13}{5} = 1{,}287$ hands with cards only of that suit. Since there are four different suits, the number of 5-card hands with cards of a single suit is $4(1{,}287) = 5{,}148$. So we have

$$
\begin{aligned}
P(\text{ All 5 cards of same suit }) &= \frac{\#\{\text{ single-suited 5-card hands }\}}{\#\{\text{ 5-card hands }\}} \\
&= \frac{5{,}148}{2{,}598{,}960} \\
&= 0.001980792.
\end{aligned}
$$