STAT 515 Lec 02 slides

Counting rules

Karl B. Gregory

University of South Carolina

Roll a die

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{\text{roll odd number}\}$

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}$$

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Motivation to study counting rules

If all outcomes in S are equally likely, for any event $A \subset S$, we have

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}.$$

This leads to an interest in counting rules.

Fundamental theorem of counting

If a job consists of K tasks such that the tasks may be completed in $n_1, \ldots, n_K$ ways, respectively, then there are

$$\prod_{k=1}^{K} n_k = n_1 \times n_2 \times \cdots \times n_K$$

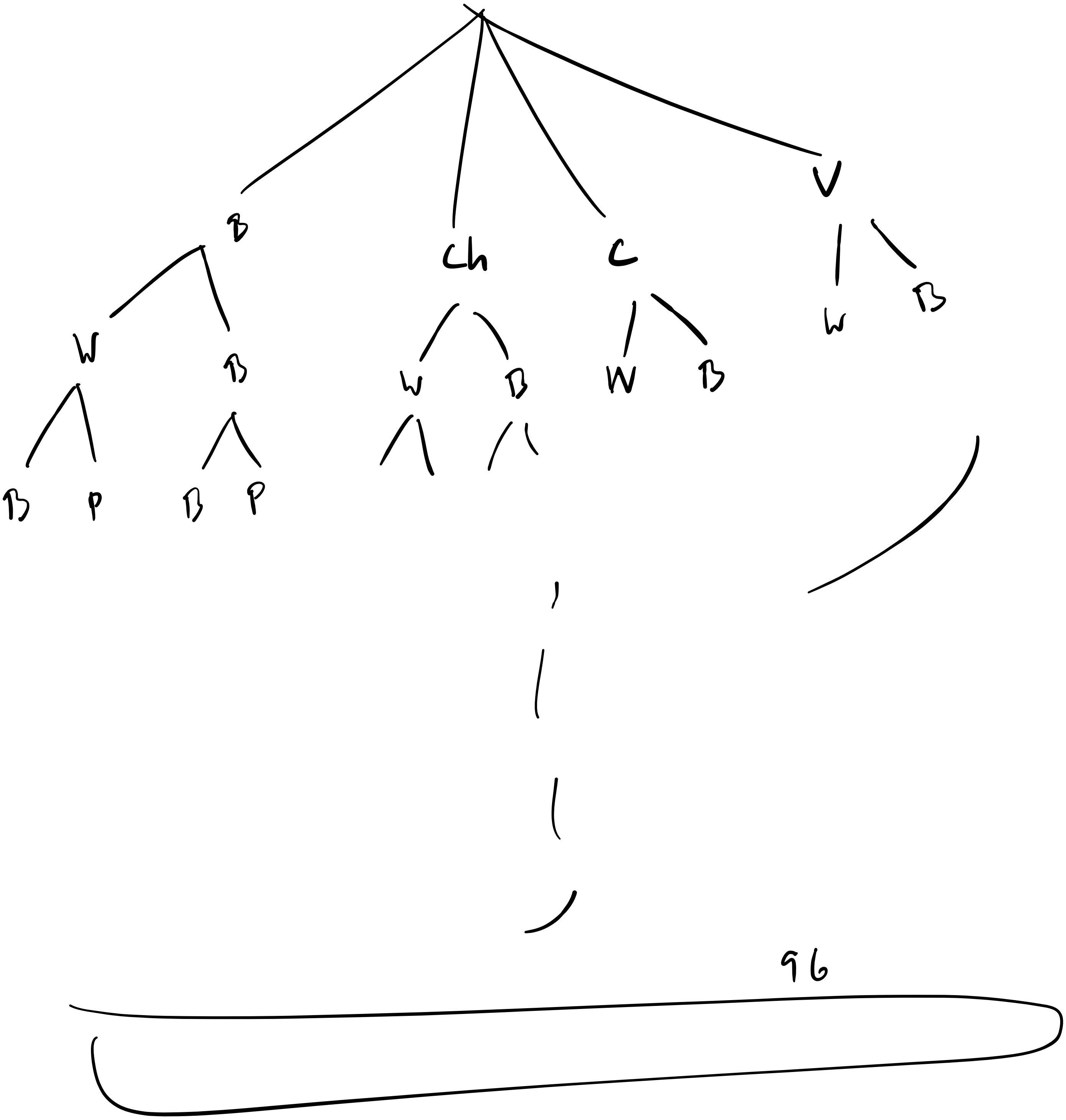
ways to do the job.

Exercise: You are confronted with the following sequence of choices:

1. Barbacoa, chicken, carnitas, or veggies $n_1 = 4$
2. White or brown rice $n_2 = 2$
3. Black or pinto beans $n_3 = 2$
4. Spicy, medium, or mild $n_4 = 3$
5. To pay extra for guacamole or not to pay extra for guacamole $n_5 = 2$

\# ways $= 4 \cdot 2 \cdot 2 \cdot 3 \cdot 2 = 96.$

In how many ways can you build your burrito?


B
Ch
C
V
W
B
W
B
W
B
W
B
B
P
B
P
96

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ $\quad \frac{7!}{(7-4)!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 7\cdot 6\cdot 5\cdot 4 = 840$

Permutation

Draw r elements of N elements without replacement and arrange them in some order. The # ways is

$$\underbrace{N(N-1)\cdots(N-r+1)} = \frac{N!}{(N-r)!}.$$

$r = 4$, $N = 7$: $7\cdot(7-1)\cdot \cdots \cdot (7-4+1) = 7\cdot 6\cdot 5\cdot 4 = 840$

Exercise: You must compose a ballad in the key of G, i.e. from the chords

G Am Bm C D Em ~~F#dim~~

1. In how many ways can four distinct chords be chosen to begin your song?
2. What if you can't play F#dim?

(2) $6\cdot 5\cdot 4\cdot 3 = 360$.

(1) $\underset{1}{\underline{7}} \cdot \underset{2}{\underline{6}} \cdot \underset{3}{\underline{5}} \cdot \underset{4}{\underline{4}} = 840$

$$\underset{1}{\underline{6}}\ \underset{2}{\underline{5}}\ \underset{3}{\underline{4}}\ \underset{4}{\underline{3}}\ \underset{5}{\underline{2}}\ \underset{6}{\underline{1}}$$

Permutation: $r = 6$, $N = 6$

$$\frac{N!}{(N-r)!} = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6!$$

Exercise: Jane Austen wrote the following novels:

- Pride and Prejudice
- Emma
- Sense and Sensibility
- Persuasion
- Northanger Abbey
- Mansfield Park

$3 \cdot 2 \cdot 1 \rightarrow$ (Pride and Prejudice, Emma, Sense and Sensibility)

6

$\boxed{0! = 1}$ $\uparrow$ Always.

1. In how many ways can you read them all this semester? $6!$
2. In how many ways can you read them all such that you read Pride and Prejudice, Emma, and Sense and Sensibility without reading any of the others in between?

$$(4 \cdot 3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1) = 6 \cdot 24 = 144.$$

Partition

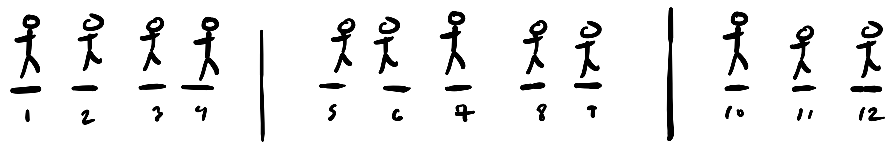
Place N balls into K urns such that the K urns receive $n_1, \ldots, n_K$ balls, where $N = n_1 + \cdots + n_N$. The # ways is

$$\frac{N!}{n_1!n_2!\cdots n_K!}.$$

$N = 12$ $K = 3$ $n_1 = 4$ $n_2 = 5$ $n_3 = 3$

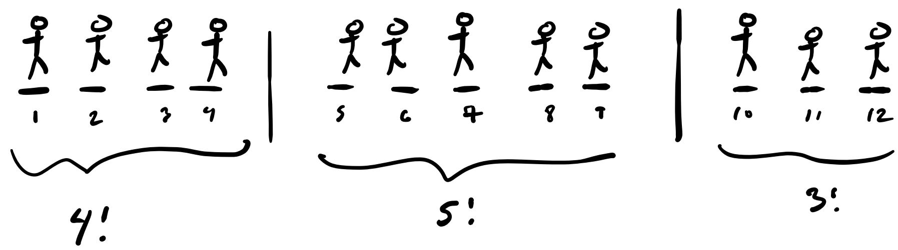
Exercise: Suppose 12 people are randomly assigned to ride in 3 vehicles taking 4, 5, and 3 passengers, respectively.

1. In how many ways can the passengers be assigned to the different vehicles?
2. If you and your friend are among these people, with what probability will the two of you ride in the same vehicle?



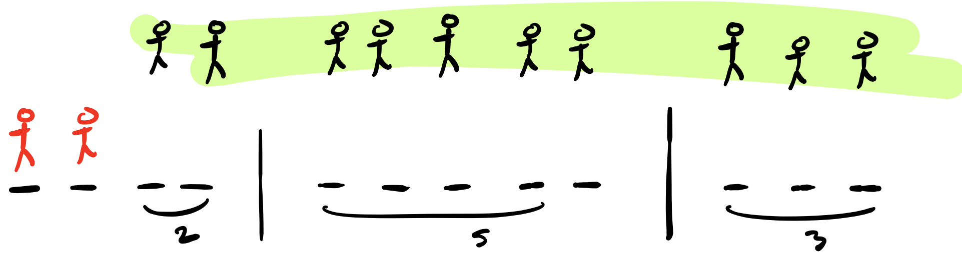
How many seating arrangements if I can about seats within vehicles?

$12!$ ways.



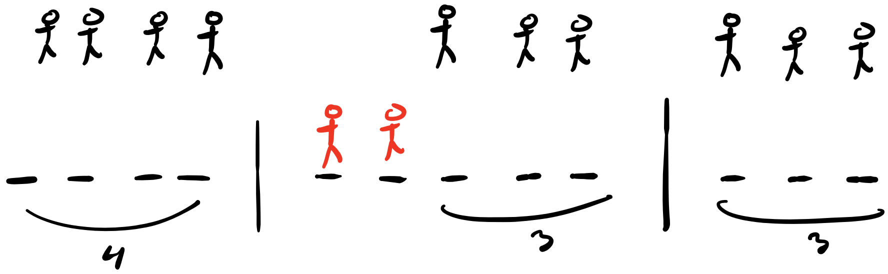
\# ways to assign them the vehicles $= \dfrac{12!}{4! \cdot 5! \cdot 3!} = \underline{27{,}720}$

$$P(\text{You ride with friend}) = \frac{\#\{\text{ways in which you ride with friend}\}}{\#\{\text{total ways to assign passengers}\}}$$

① 

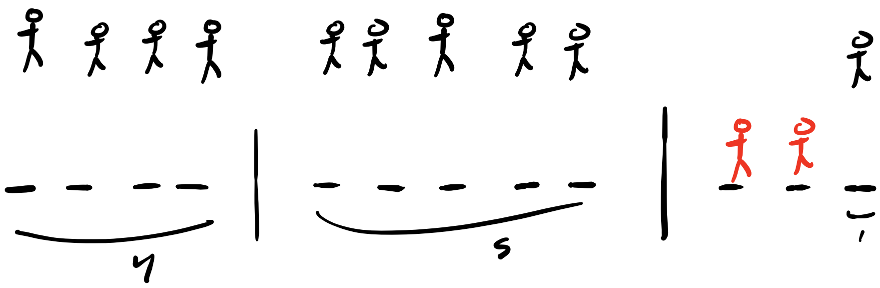
$= \dfrac{10!}{2!\ 5!\ 3!} = 2{,}520$

2



$$= \frac{10!}{4!\,3!\,3!} = 4200$$

3



$$= \frac{10!}{4!\,5!\,1!} = 1260$$

$$P\left(\text{You + Friend together}\right) = \frac{2520 + 4200 + 1260}{27720} = 0.2879.$$

$$\frac{6!}{2!\,4!} = \frac{6!}{2!(6-2)!} = \binom{6}{2} \overset{?}{=} \binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!\,2!}$$

Combination

Draw r elements from N without replacement and without regard to their order. The # ways is

$$\underbrace{\binom{N}{r}}_{\text{"N choose r"}} = \frac{N!}{r!(N-r)!}.$$

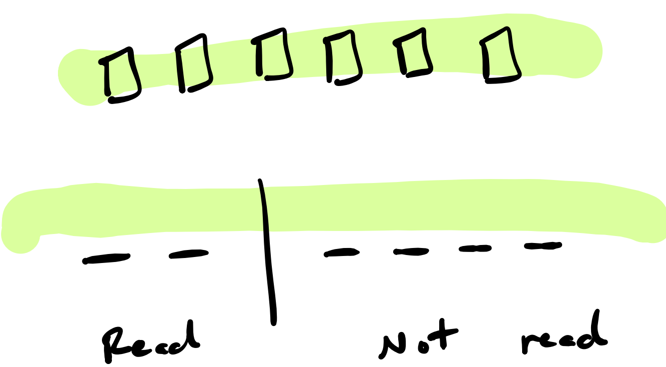
$$\frac{6!}{2!(6-2)!} = \frac{6!}{2!\,4!} = 15$$

"N choose r" R: choose(N, r)

This is just a partition with only 2 "urns".

$N=6$ $r=2$

Exercise: In how many ways could you choose 2 of the 6 J.A. novels to read?



$$\frac{6!}{2!\,4!} = \frac{6\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{2\cdot1\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 15$$

5 episodes

10	BR
8	BD
12	HC

Exercise: For German class you must watch 5 episodes from among 10 available episodes of Bares für Rares, 8 of Betty's Diagnose, and 12 of Haustier Check.

1. In how many ways can you choose
   a) 5 episodes of Bares für Rares? $\binom{10}{5} = \frac{10!}{5!\,5!} = 252$
   b) 3 episodes of Haustier Check and 2 episodes of Betty's Diagnose? $\binom{12}{3}\cdot\binom{8}{2} = 6160$
2. If you choose 5 episodes at random, with what probability do you
   a) not watch any episodes of Haustier Check?
   b) watch at least one episode of Bares für Rares?
   c) binge entirely on Betty's Diagnose?

$$P(a) = \frac{\#\{\text{ways choose 5 but none of HC}\}}{\#\{\text{ways to choose 5 episodes}\}} = \frac{\binom{18}{5}}{\binom{30}{5}}$$

$$P(b) = 1 - P(\text{no episodes of BF}) = 1 - \frac{\binom{20}{5}}{\binom{30}{5}}$$

$$P(C) = \frac{\binom{8}{5}}{\binom{30}{5}}$$

Exercise: If dealt 5 cards from a 52-card deck, what is the probability of getting

1. the ace of diamonds?
2. at least one ace?

$$\binom{1}{1} = \frac{1!}{1!(1-1)!} = 1$$

① $$P(A\diamondsuit) = \frac{\#\{\text{5-card hands with } A\diamondsuit\}}{\#\{\text{5-card hands}\}}$$

task 1: put $A\diamondsuit$ in hand. $\binom{1}{1}$

$$= \frac{\binom{1}{1}\binom{51}{4}}{\binom{52}{5}}$$

task 2: get 4 more cards $\binom{51}{4}$

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}$$

$$= \frac{\frac{1!}{1!(1-1)!} \quad \frac{51!}{4!\underbrace{(51-4)}!}}{\frac{52!}{5!\underbrace{(52-5)}!}}$$

$$= \frac{51!}{4!\ \cancel{47!}} \cdot \frac{5!\ \cancel{47!}}{52!}$$

$$= \frac{51!}{52!} \quad \frac{5!}{4!}$$

$$= \frac{\cancel{51!}}{52 \cdot \cancel{51!}} \quad \frac{5 \cdot \cancel{4!}}{\cancel{4!}}$$

$$= \frac{5}{52}$$

$$= P(A\diamondsuit)$$

② $P(\text{At least one Ace}) = 1 - P(\text{No Aces})$

$$= 1 - \frac{\#\{\text{hands with no Aces}\}}{\#\{\text{5-card hands}\}}$$

$$= 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

$$= 1 - \frac{\frac{48!}{5!(48-5)!}}{\frac{52!}{5!(52-5)!}}$$

$$= 1 - \frac{48!}{\cancel{5!}\,43!} \quad \frac{\cancel{5!}\,47!}{52!}$$

$$= 1 - \frac{47 \cdot 46 \cdot 45 \cdot 44}{52 \cdot 51 \cdot 50 \cdot 49}$$

$$= 0.341.$$

4. Suppose you draw 1 athlete at random from a group of 100 athletes such that: 30 swim; 44 run; 9 swim and run; 5 swim, bike, and run; 11 swim and bike; 10 bike and run but do not swim; and 35 only bike. Let S, B, and R denote the events that the athlete you draw swims, bikes, and runs, respectively. Give the following probabilities:
   (a) $P(S \cup R)$
   (b) $P(S \cap R^c)$
   (c) $P(B)$
   (d) $P(S \cup B)$
   (e) $P((S \cap R) \cap B^c)$
   (f) $P(S^c \cup R^c)$
   (g) $P((R \cap B) \cup (R \cap B^c))$