STAT 515 Lec 08 slides

The Poisson and Exponential distributions

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Jellyfish

x	0	1	2	3	4	5	......
$P(X \leq x)$							

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Poisson process

Suppose $X = \#$ occurrences per unit of time or space, where the occurrences

1. are independent
2. occur randomly but at a constant rate over the entire time/space.

A process generating such occurrences is called a *Poisson process*.

Examples:

1. # customers entering a store in an hour.
2. # earthquakes per decade in a region.
3. # weeds growing per square meter of a field.
4. # bird nests per acre in a habitat.

λ = "lambda"

probability mass function

Poisson distribution

The probability distribution with pmf given by

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, \ldots$$

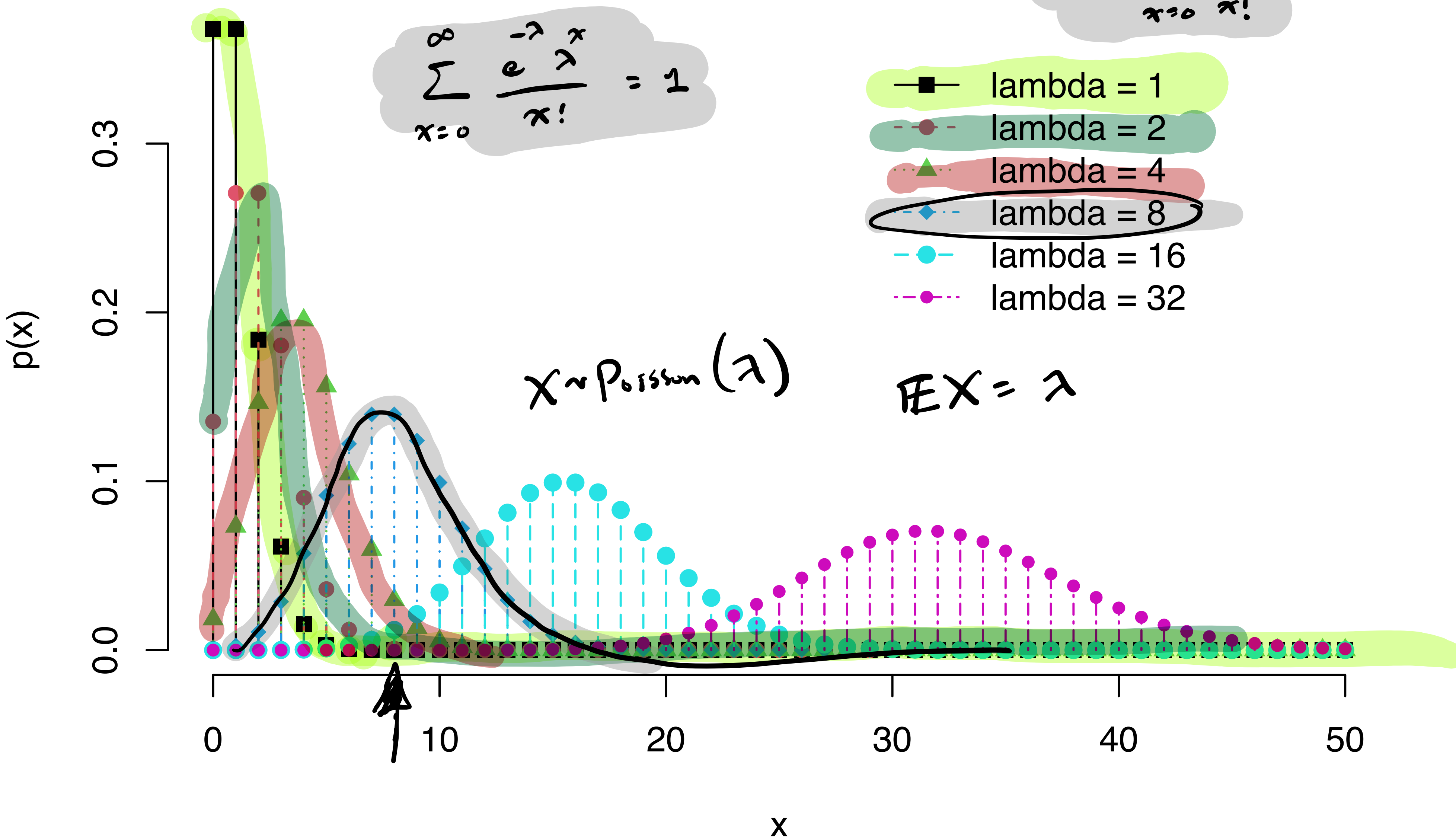
with $\lambda > 0$ is called the *Poisson distribution*.

We write $X \sim \text{Poisson}(\lambda)$.

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots$$

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$$



$X \sim \text{Poisson}(\lambda)$

$EX = \lambda$

Mean and variance of Poisson distribution

If $X \sim \text{Poisson}(\lambda)$ then

- $\mathbb{E}X = \lambda$.
- $\text{Var}\, X = \lambda$.

Exercise: Let $X \sim \text{Poisson}(20)$ be the # car accidents in town on a given day.

1. Find $P(X = 10)$.
2. Find the probability that there is at least one accident.
3. Find $P(X \leq 15)$.
4. Find $P(X \geq 20)$.
5. If X observed on many days, to what will the average of the values be close?

Introduce `dpois` and `ppois` functions in R.

$X \sim \text{Poisson}(\lambda = 20)$

$$p(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \qquad x = 0, 1, 2, \ldots \qquad \text{pmf of Poisson}$$

① $P(X = 10) = \dfrac{e^{-20} (20)^{10}}{10!} = 0.0058 = \text{dpois}(10, \text{lambda} = 20)$

② $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - \frac{e^{-20} (20)^{0}}{0!}$$

$$= 1 - e^{-20}$$

$$\approx 1$$

③ $P(X \leq 15) = P(X = 0) + P(X = 1) + \cdots + P(X = 15)$

$$= \sum_{x=0}^{15} P(X = x)$$

$$= \sum_{x=0}^{15} \frac{e^{-20} (20)^{x}}{x!}$$

$$= \text{ppois}(15, \text{lambda} = 20) \quad (\text{cdf of Poisson})$$

$$= 0.1565$$

④ $P(X \geq 20) = P(X = 20) + P(X = 21) + \cdots$

$$= \sum_{x=20}^{\infty} P(X = x)$$

$$= \sum_{x=20}^{\infty} \frac{e^{-20}(20)^x}{x!}$$

$$= 1 - P(X < 20)$$

$$= 1 - P(X \le 19)$$

$$= 1 - \sum_{x=0}^{19} \frac{e^{-20}(20)^x}{x!}$$

$$= 1 - \text{ppois}(19, \text{lambda} = 20)$$

$$= 0.5297.$$

Mean number of occurrences scales with unit of time/space...

Let $X \sim \text{Poisson}(\lambda)$, where $X = \#$ occurrences per unit time/space of an event.

Then if $Y = \#$ occurrences in t units of time/space, we have $Y \sim \text{Poisson}(t\lambda)$.

X = # accidents in 1 day ~ Poisson (20)

Exercise: Let Y be the # car accidents in town in a given week.

1. What is the distribution of Y? Refer to previous example.
2. Find $P(Y \leq 130)$.
3. Find $P(Y = 140)$.
4. Find $P(Y \geq 150)$.

$Y \sim \text{Poisson}(\underbrace{7 * 20}_{140})$, $\mathbb{E}Y = 140$

ppois(130, lambda = 140) = 0.2124

Consider the time between occurrences in a Poisson process...

Exponential distribution

The continuous probability distribution with pdf and cdf given by

probability density function $\longrightarrow$ $f(y) = \lambda e^{-y\lambda}$ $\longleftarrow$ pdf

cumulative dist. function $\longrightarrow$ $F(y) = 1 - e^{-y\lambda}$ for $y > 0$ $\quad P(Y \leq y)$

with $\lambda > 0$ is called the *Exponential distribution*.

We write $Y \sim \text{Exponential}(\lambda)$.

Derive: Start with $P(Y > y) = P(\text{no occurrences before time } y)$.

Let Y = time until 1^{st} event. Get $F(y) = P(Y \leq y)$ (cumulative dist. function)

$X \sim \text{Poisson}(\lambda)$.

For any elapsed time $y > 0$, we have

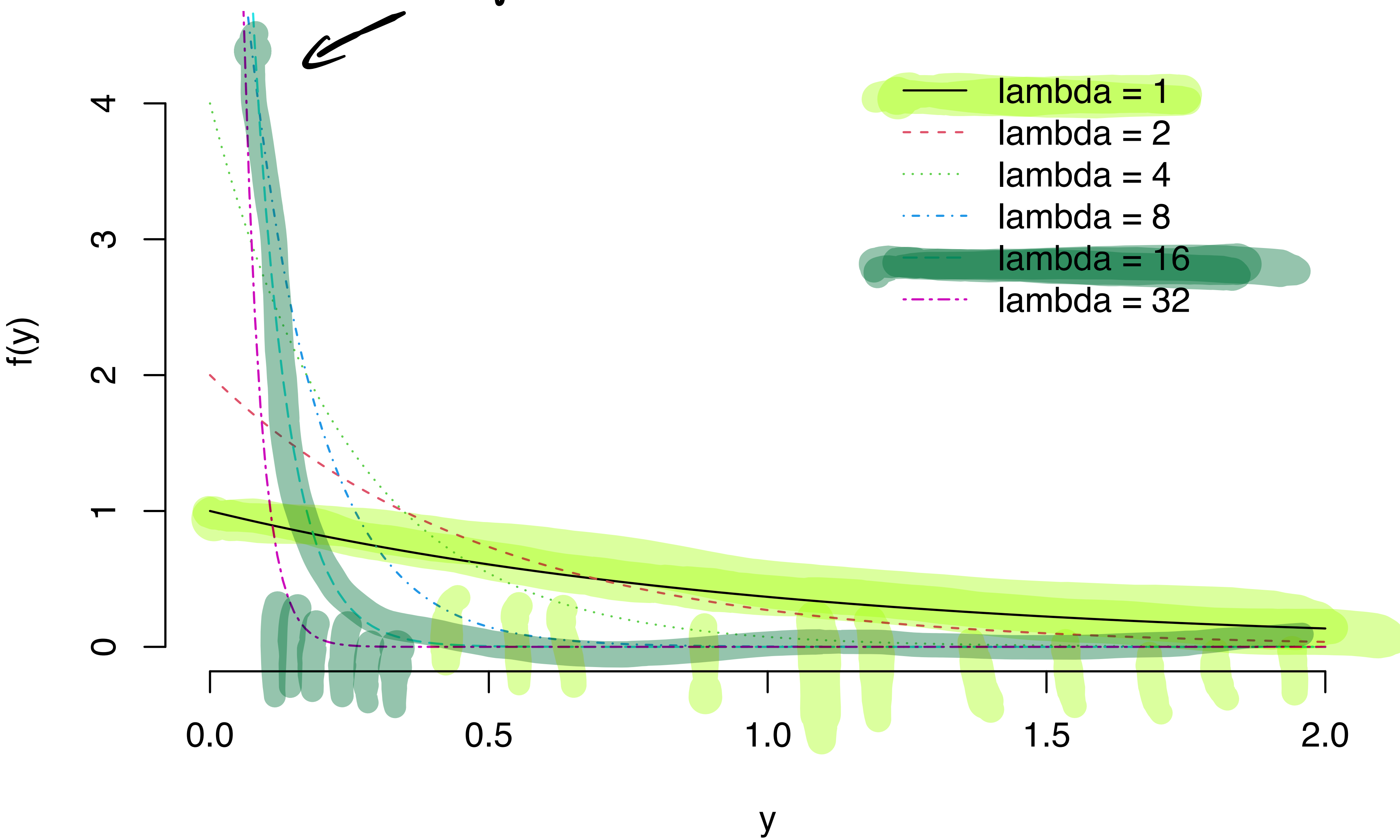
$$
\begin{aligned}
F(y) &= P(Y \leq y) \\
&= 1 - P(Y > y) \\
&= 1 - P(\text{No events before time } y) \\
&= 1 - P(X_y = 0) \qquad X_y \sim \text{Poisson}(y\lambda) \\
&= 1 - \frac{e^{-y\lambda}(y\lambda)^0}{0!} \\
&= 1 - e^{-y\lambda}
\end{aligned}
$$

rescale the time interval

$$F(y) = P(Y \leq y) = \int_0^y f(y')\,dy'$$

$$\Rightarrow \quad f(y) = \frac{d}{dy} F(y) = \lambda e^{-y\lambda}.$$

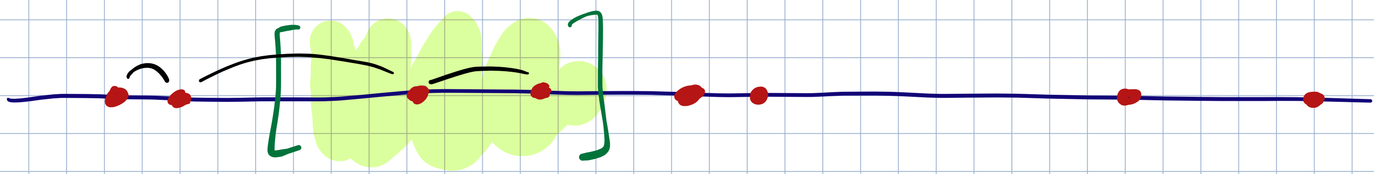
pdfs of $Y \sim \text{Exponential}(\lambda)$.

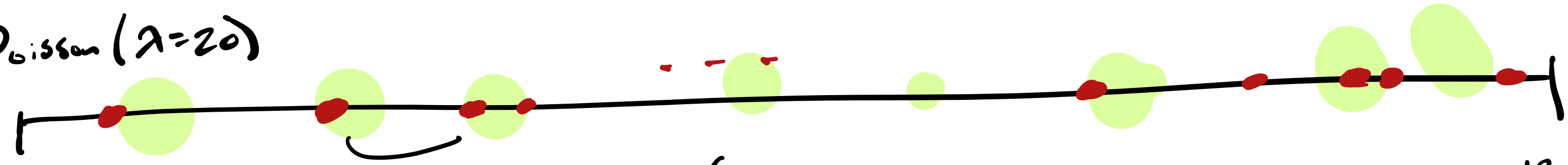


Poisson process

X = # events in one unit in time/space $\sim$ Poisson (λ) $\quad \mathbb{E}X = \lambda$

Y = Amount of time until next event $\sim$ Exponential (λ) $\quad \mathbb{E}Y = \frac{1}{\lambda}$





Mean and variance of Exponential distribution

If $Y \sim \text{Exponential}(\lambda)$ then

- $\mathbb{E}Y = 1/\lambda$.
- $\text{Var } Y = 1/\lambda^2$.

$$F(y) = 1 - e^{-y\lambda}$$

$X \sim \text{Poisson}(20)$

Exercise: Suppose the occurrence of car accidents in a town is a Poisson process with the expected number of accidents per day equal to 20.

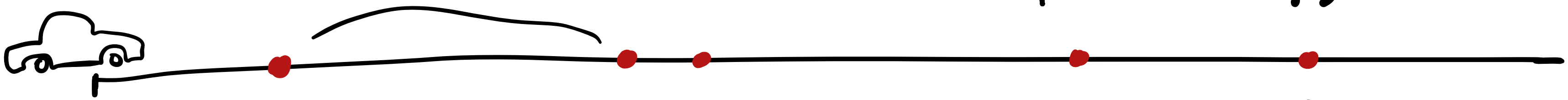
1. What is the expected time between car accidents? $\mathbb{E}Y = \frac{1}{20}$
2. What is the probability that an accident happens in the next hour?

② Y = time until next accident.

$$P(Y \le 1\text{hr}) = P\left(Y \le \tfrac{1}{24} \text{ days}\right)$$

$$= 1 - e^{-\frac{1}{24}20} = 1 - e^{-\frac{5}{6}} = 0.5654$$

Y = distance until next tire $\sim$ Exponential$(\lambda = \frac{1}{3})$



X = # tires in 1 mile $\sim$ Poisson$(\lambda = \frac{1}{3})$

Exercise: Suppose the occurrence of blown-out tires along a freeway is a Poisson process with the expected number of blown-out tires per mile equal to $1/3$.

Find the probabilities of the following:

Poisson pmf.

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

1. finding 2 blown-out tires in the next mile.
2. finding at least one blown-out tire in the next mile.
3. finding fewer than 3 blown-out tires in the next 12 miles.
4. finding a blown-out tire before going 5 miles.
5. not finding a blown-out tire in the next 3 miles.
6. finding a blown-out tire exactly 3 miles down the road.

① $P(X=2) = \dfrac{e^{-\frac{1}{3}}(\frac{1}{3})^2}{2!} =$ dpois(2, lambda = 1/3) $= 0.0398$

② $P(X \geq 1) = 1 - P(X=0) = 1 - \dfrac{e^{-1/3}(\frac{1}{3})^0}{0!} = 1 - e^{-\frac{1}{3}} = 0.2835$

(3) Let W = # tires in 12 miles.

Then $W \sim \text{Poisson}(\lambda = \underbrace{12 \cdot \tfrac{1}{3}}_{4})$

1 - dpois(0, lambda = 1/3)

$P(W < 3) = P(W \leq 2) =$ ppois(2, lambda = 4) $= 0.2381.$

$$\boxed{\begin{aligned} &= P(W=0) + P(W=1) + P(W=2) \\ &= \sum_{w=0}^{2} P(W=w) \\ &= \sum_{w=0}^{2} \frac{e^{-4}\, 4^{w}}{w!} \end{aligned}}$$

(4) Y = distance till next tire

$F(y) = 1 - e^{-y\lambda}$

$$P(Y < 5) = P(Y \leq 5)$$
$$= 1 - e^{-5\left(\frac{1}{3}\right)}$$
$$= 1 - e^{-5/3}$$
$$= 0.811$$

5 not finding a blown-out tire in the next 3 miles.

$$P(Y > 3) = 1 - P(Y \leq 3)$$
$$= 1 - \left(1 - e^{-3\left(\frac{1}{3}\right)}\right)$$

$= e^{-1}$

$= 0.3679$