STAT 515 fa 2021 Exam I

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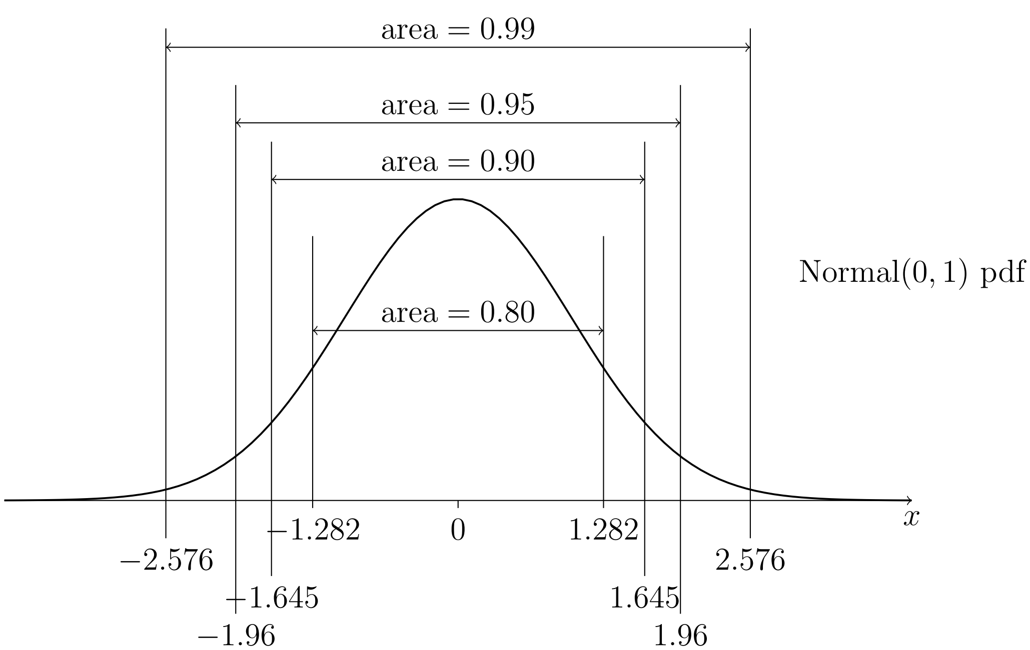
- Do not open this exam until told to do so.
- You may have one handwritten sheet of notes out during the exam.
- You have 75 minutes to work on this exam.
- You may NOT use any kind of calculator.
- If you are unsure of what a question is asking for, do not hesitate to ask me for clarification.
- *Good luck, and may the odds be ever in your favor!*

$X \sim$		$\mathcal{X}$	$\mathbb{E}X$	$\text{Var}(X)$
Binomial(n, p)	$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, 1, \dots, n$	np	$np(1-p)$
Poisson(λ)	$P(X = x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$	$x = 0, 1, 2, \dots$	λ	λ
Exponential(λ)	$P(X \leq x) = 1 - e^{-x\lambda}$	$x > 0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$



1. Circle "True" or "False" for each of the following statements:
   (a) True **FALSE** A larger sample size increases the variance of the sample mean.
   (b) **TRUE** False Independence of two events implies that they are not mutually exclusive.
   (c) True **FALSE** If X is the number of successes in a number of independent Bernoulli trials, then X has a hypergeometric distribution.
   (d) True **FALSE** Every random variable has a probability density function.
   (e) True **FALSE** The support of a random variable cannot contain negative numbers.

2. Suppose 5% of a population has a genetic mutation which causes individuals to develop, with probability 0.30, a type of cancer, whereas it is expected that 90% of individuals without the mutation will not develop the cancer.
   (a) What proportion of those with the cancer would you expect to possess the genetic mutation?

Let M be the event that an individual has the mutation and C be the event that the individual develops the cancer. Then it is given that $P(M) = 0.05$, $P(C|M) = 0.30$ and $P(C^c|M^c) = 0.90$. By Bayes' rule we have

$$\begin{aligned} P(M|C) &= \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C^c|M^c)P(M^c)} \\ &= \frac{(0.30)(0.05)}{(0.30)(0.05) + (1-0.90)(1-0.05)} \\ &= 0.1363636 = 3/22. \end{aligned}$$

   (b) What proportion of individuals in the whole population would you expect to develop the cancer?

This is equal to the denominator of the above, which is

$$P(C) = P(C|M)P(M) + P(C|M^c)P(M^c) = (0.30)(0.05) + (1-0.90)(1-0.05) = 0.11.$$

3. On average, 15 pilgrims per day visit a shrine, with all days and times of day being alike.

   (a) Suggest a probability distribution for the number of pilgrims visiting the shrine on a given day.

   The Poisson($\lambda = 15$) distribution.

   (b) Suggest a probability distribution for the number of pilgrims visiting the shrine in a given week.

   The Poisson($\lambda = 105$) distribution, where $105 = 7 \cdot 15$.

   (c) Give an expression for the probability that fewer than 30 pilgrims visit the shrine in the next two days (you do not need to evaluate the expression).

   The number of pilgrims coming over a two-day time period would have a Poisson(30) distribution, where $30 = 2 \cdot 15$. So we would compute this probability as

$$\sum_{x=0}^{29} \frac{e^{-30}(30)^x}{x!} = \texttt{ppois(29,30)} = 0.475717.$$

   (d) If each pilgrim, on arriving at the shrine, lights a candle which burns for one hour, with what probability will the candle be burning when the next pilgrim arrives?

   Let Y be the amount of time until the next pilgrim arrives. Then $Y \sim$ Exponential(15), where the unit of time is one day. We are interested in the event that $Y \leq 1/24$, since there are 24 hours in a day. So we have

$$P(Y < 1/24) = 1 - e^{-(1/24)15} = 1 - e^{-5/8} = 0.4647386.$$

   We have used the cdf of the Exponential(λ) distribution, which is given by $F(y) = 1 - e^{y\lambda}$.

4. A variety of asparagus produces spears with diameters having the Normal distribution with mean $\mu = 13$mm and standard deviation $\sigma = 1.5$mm. A packing house that prepares bundles of spears to be sold in supermarkets wants no more than 2% of the spears to have diameter less than 10mm.

   (a) Find the proportion of spears from this variety with diameter less than 10mm.

   Letting $X \sim \text{Normal}(\mu = 13, \sigma^2 = (1.5)^2)$, we write

   $$P(X < 10) = P((X - 13)/1.5 < (10 - 13)/1.5) = P(Z < -2) = 0.5 - 0.4772 = 0.0228.$$

   (b) Give an interval, centered at the mean, within which 90% of the spear diameters lie.

   We must go 1.645 standard deviations above and below the mean. So the interval is

   $$13 \pm 1.645(1.5) = (10.5325, 15.4675).$$

   (c) Suppose the mean spear diameter can be increased by amending the soil. To what must μ be increased in order that no more than 2% of the spears will have diameters less than 10mm?

   We begin by writing $P(X < 10) \leq 0.02$. From here, we have

   $$\begin{aligned} P(X < 10) \leq 0.02 &\iff P((X - 13)/1.5 < (10 - \mu)/1.5) \leq 0.02 \\ &\iff P(Z < (10 - \mu)/1.5 \leq 0.02 \\ &\iff (10 - \mu)/1.5 \geq q^Z_{0.02}. \end{aligned}$$

   We have $q^Z_{0.02} = -2.05$ (by finding 0.4798 on the z-table). Plugging this in and rearranging the above expression gives

   $$\mu \geq 10 - 1.5(-2.05) = 13.075.$$

   (d) If the packing house samples 4 spears at random, with what probability will the mean of their diameters be less than 11mm? (Use $\mu = 13$mm).

   We have $\bar{X}_4 \sim \text{Normal}(13, (1.5)^2/4)$. So we have

   $$\begin{aligned} P(\bar{X}_4 < 11) &= P((\bar{X}_5 - 13)/(1.5/2) < (11 - 13)/(1.5/2)) \\ &= P(Z < -2.67) \\ &= 0.5 - 0.4962 \\ &= 0.0038. \end{aligned}$$

5. A crate of asparagus arrives at a packing house. It contains 500 spears, 30 of which have diameter less than 10mm. The packing house quality control worker samples 20 spears. Give mathematical expressions for the following (you do not need to evaluate them):

(a) If he samples with replacement,

i. the probability that two or more of the sampled spears have diameter less than 10mm.

Letting X be the number of sampled spears with diameter less than 10mm, we have $X \sim \text{Binomial}(20, 30/500)$. So we have

$$P(X \geq 2) = \sum_{x=2}^{20} \binom{20}{x} (30/500)^x (1 - 30/500)^{20-x} = \texttt{1-pbinom(1,20,30/500)} = 0.3395454.$$

ii. the probability that none of the sampled spears has diameter less than 10mm.

$$P(X = 0) = (1 - 30/500)^{20} = 0.2901062.$$

(b) If he samples without replacement,

i. the probability that two or more of the sampled spears have diameter less than 10mm.

Letting X be the number of sampled spears with diameter less than 10mm, we have $X \sim \text{Hypergeometric}(N = 500, M = 30, K = 20)$. So we have

$$P(X \geq 2) = \sum_{x=2}^{20} \frac{\binom{30}{x}\binom{500-30}{20-x}}{\binom{500}{20}} = \texttt{1-phyper(1,30,500-30,20)} = 0.3405865.$$

ii. the probability that none of the sampled spears has diameter less than 10mm.

We have

$$P(X = 0) = \frac{\binom{30}{0}\binom{500-30}{20-0}}{\binom{500}{20}} = 0.2829643.$$

(c) Comment on whether the answers to parts (a) and (b) would be close to each other or not (if you were to compute them), and say why.

Since the number of asparagus spears is large (the "bag of marbles" or the population), a draw without replacement does not change its composition by much, so that the difference between drawing with and without replacement is negligible; the answers will be very close to each other.