STAT 515 sp 2024 Exam I

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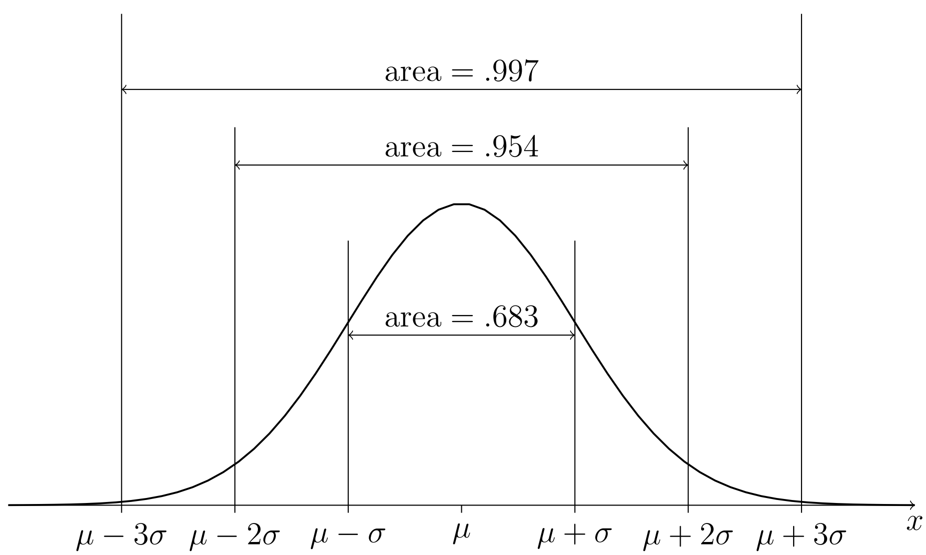
- Do not open this exam until told to do so.
- You may have one handwritten sheet of notes out during the exam.
- You have 75 minutes to work on this exam.
- You may NOT use any kind of calculator.
- If you are unsure of what a question is asking for, do not hesitate to ask me for clarification.
- *Good luck, and may the odds be ever in your favor!*

$X \sim$		$\mathcal{X}$	$\mathbb{E}X$	$\text{Var}(X)$
Binomial(n,p)	$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$x=0,1,\ldots,n$	np	$np(1-p)$



1. Among the patrons of a library, 80% are at least thirty years old. Those at least thirty years old borrow a hard-copy book 70% of the time and an ebook 30% of the time. Those younger than thirty borrow a hard-copy book 40% of the time an ebook 60% of the time.

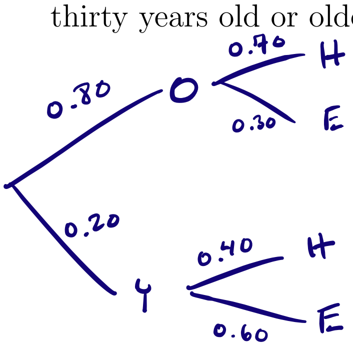
   (a) Give the probability that the next book borrowed by a randomly selected patron is a hard-copy book.

   Let H = hard copy, O = ≥ 30 years old, Y = < 30 yrs old.

   Then $P(H) = P(H|O)P(O) + P(H|Y)P(Y) = (0.70)(0.80) + (0.40)(0.20)$

   $= 0.56 + 0.08 \boxed{= 0.64}$

   (b) If a randomly selected patron borrows a hard-copy book, give the probability that the patron was thirty years old or older.



   $$P(O|H) = \frac{P(O \cap H)}{P(H)} = \frac{P(H|O)P(O)}{P(H)} = \frac{(0.7)(0.8)}{0.64}$$

   $$= \frac{0.56}{0.64}$$

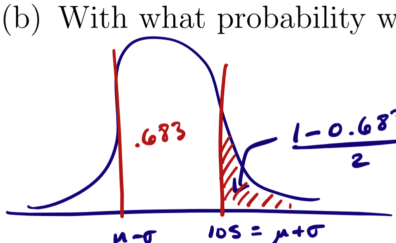
   $$\boxed{= \frac{7}{8}}.$$

2. A grower of Pink Lady apples brings to market apples weighing, on average, 100 grams. ~~The standard deviation of the apple weights is 5 grams.~~ Suppose the standard deviation of the apple weights is 5 grams and that the weights have a Normal distribution.

   (a) What proportion of the apples have weights between 90 and 110 grams?

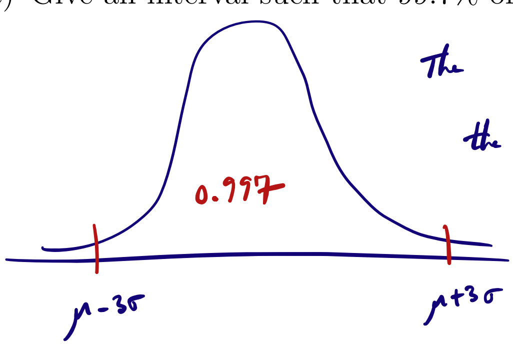
   This is within 2 standard deviations of the mean.
   The diagram tells us the area under the Normal pdf on this interval is $\boxed{0.954.}$

   (b) With what probability would a randomly selected apple weigh more than 105 grams?



   $$\frac{1-0.683}{2} = \frac{0.317}{2} \boxed{= 0.1585.}$$

   (c) Give an interval such that 99.7% of apples from this grower would have a weight in the interval.



   The interval $[85, 115]$, which is centered at the mean and extending three standard deviations above and below it, contains the weights of 99.7% of the apples.

3. Consider the phrase *all mimsy were the borogoves.*

(a) How many sequences of words can you make by rearranging the words in the phrase?

We can make $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$ sequences of words.

(b) In a random rearrangement, with what probability will *borogoves* be one of the first two words?

There are $4!$ arrangements in which "borogoves" is the 1^{st} word, and another $4!$ arrangements in which it is the 2^{nd} word. So $\frac{4!}{5!} + \frac{4!}{5!} = \frac{1}{5} + \frac{1}{5} = \boxed{\frac{2}{5}}$ is the probability.

(c) How many unique sequences of 5 letters can you make by rearranging the letters in *mimsy*?

If all the letters were unique, the answer would be $5!$, but since there are two "m"s, the number $5!$ counts duplicates: we must divide by $2!$, so the answer is $\boxed{\frac{5!}{2!} = 60}$.

(d) In how many ways can you choose two words in the phrase to cross out?

This is $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = \boxed{10}.$

(e) In how many ways can you choose three words in the phrase to cross out?

This is also $\boxed{10}$, since choosing 2 to cross out is the same as choosing 3 to <u>not</u> cross out.

4. Suppose a breed of dog has litter sizes 1,2,...,7 with the probabilities given in the table:

litter size	1	2	3	4	5	6	7
probability	0.1	0.2	0.3	0.2	0.1	0.05	0.05

(a) Give the probability of a litter size of at least 2 puppies.

Let X = litter size. Then

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X=1) = 1 - 0.10 = \boxed{0.90}.$$

(b) Give a table showing the cumulative probabilities for the litter sizes, that is $P(X \leq x)$, for each $x = 1, 2, \dots, 7$, where X is the litter size.

x	1	2	3	4	5	6	7
$P(X \leq x)$	0.1	0.3	0.6	0.8	0.9	0.95	1.00

(c) Give the expected value of the litter size.

We have

$$\mathbb{E}X = 1(0.1) + 2(0.2) + 3(0.3) + 4(0.2) + 5(0.1) + 6(0.05) + 7(0.05)$$

$$= 0.1 + 0.4 + 0.9 + 0.8 + 0.5 + 0.30 + 0.35 = $$

$$\begin{array}{r} 2.40 \\ 0.30 \\ 0.35 \\ \hline 3.35 \end{array}$$

So $\boxed{\mathbb{E}X = 3.35}$

5. For three applicants to a graduate program, let A_1, A_2, and A_3 be the events that the applicants are accepted. Express the following events using elementary set operations on A_1, A_2, and A_3.

   (a) At least one of the applicants is accepted.

   $$A_1 \cup A_2 \cup A_3$$

   (b) None of the applicants is accepted.

   $$(A_1 \cup A_2 \cup A_3)^c = A_1^c \cap A_2^c \cap A_3^c$$

   (c) Exactly two of the applicants are accepted.

   $$(A_1 \cap A_2 \cap A_3^c) \cup (A_1 \cap A_2^c \cap A_3) \cup (A_1^c \cap A_2 \cap A_3)$$

6. Suppose a six-sided die is rolled five times. Let X be the number of ⚅'s rolled.

   (a) What is the name of the probability distribution of X?

   $X \sim \text{Binomial}(n=5,\ p=1/6)$, the Binomial distribution.

   (b) Give an expression (you do not need to evaluate it) for $P(X = 3)$.

   $$P(X=3) = \binom{5}{3}\left(\tfrac{1}{6}\right)^3\left(1-\tfrac{1}{6}\right)^{5-3}$$

   (c) Give the probability that you will roll all ⚅'s.

   $$P(X=5) = \left(\tfrac{1}{6}\right)^5$$

   (d) Give the expected value of X.

   $$\mathbb{E}X = 5\left(\tfrac{1}{6}\right) = \tfrac{5}{6}.$$