STAT 515 sp 2024 Final Exam

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- Do not open this exam until told to do so.
- You may have two handwritten sheets of notes out during the exam.
- You have 150 minutes to work on this exam.
- You may NOT use any kind of calculator.
- If you are unsure of what a question is asking for, do not hesitate to ask me for clarification.
- *Good luck, and may the odds be ever in your favor!*

$X \sim$		$\mathcal{X}$	$\mathbb{E}X$	$\text{Var}(X)$
Binomial(n,p)	$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, 1, \ldots, n$	np	$np(1-p)$
Poisson(λ)	$P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$	$x = 0, 1, 2, \ldots$	λ	λ
Exponential(λ)	$P(X \leq x) = 1 - e^{-x\lambda}$	$x > 0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$

$$\hat{p}_n \pm z_{\alpha/2} \cdot \sqrt{\hat{p}_n(1-\hat{p}_n)/n} \qquad \bar{X}_n \pm t_{n-1,\alpha/2} \cdot S_n/\sqrt{n}$$

$$Z_{\text{test}} = \frac{\hat{p}_n - p_0}{\sqrt{p_0(1-p_0)/n}} \qquad T_{\text{test}} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

A t-table is attached to this exam.

1. Eighty percent of purchasers of a 14" carbon steel wok from KG's discount store are satisfied with the wok. Ten percent of the purchasers post a review of the wok on the store's website. Of the reviews, ninety percent indicate satisfaction with the wok. Give the probability that:

   (a) A satisfied purchaser posts a review of the wok.

S = satisfied, R = posts review

$P(S) = \frac{8}{10}$ $\quad P(R) = \frac{1}{10}$ $\quad P(S|R) = \frac{9}{10}$

We want $P(R|S) = \frac{P(R \cap S)}{P(S)} = \frac{P(S|R)P(R)}{P(S)} = \frac{\frac{9}{10}\,\frac{1}{10}}{\frac{8}{10}} = \boxed{\frac{9}{80}}$

   (b) A customer who does not post a review is satisfied with the wok.

$$P(S|R^c) = \frac{P(S \cap R^c)}{P(R^c)} = \frac{P(R^c|S)P(S)}{P(R^c)} = \frac{[1 - P(R|S)]P(S)}{P(R^c)}$$

$$= \frac{\left(1 - \frac{9}{80}\right)\frac{8}{10}}{1 - \frac{1}{10}}$$

$$= \frac{\left(\frac{71}{80}\right)\frac{8}{10}}{\frac{9}{10}}$$

$$= \boxed{\frac{71}{90}}$$

2. Ten ~~purchasers~~ people order a 14" carbon steel wok from KG's discount store. Remember that 80% of purchasers are satisfied with the wok. Assuming each purchaser's satisfaction or dissatisfaction with the wok to be independent of that of the other purchasers, give an expression for the probability that:

(a) Exactly 8 purchasers are satisfied with the wok. (you do not have to evaluate it)

Let X = # purchasers satisfied. Then $X \sim \text{Binomial}(n = 10, p = 0.80)$

$$P(X = 8) = \binom{10}{8} (0.80)^{8} (1 - 0.80)^{10-8}$$

(b) All 10 purchasers are satisfied with the wok.

$$P(X = 10) = \binom{10}{10} (0.80)^{10} (1 - 0.80)^{10-10} = (0.80)^{10}$$

(c) At least one purchaser is dissatisfied with the wok.

This means $X \leq 9$.

We have $P(X \leq 9) = 1 - P(X = 10) = 1 - (0.80)^{10}$

3. In order to season the 14" carbon steel wok for first-time use, purchasers must spend several minutes super-heating it until the metal acquires a bluish tint. Nine randomly selected purchasers of the wok recorded in a survey the number of minutes they spent super-heating their woks. The mean and standard deviation of the reported numbers of minutes were $\bar{X}_n = 35$ and $S_n = 5$. Mr. KG of KG's discount store holds firmly to the conviction that properly seasoning a wok requires at least 40 minutes of super-heating. If he concludes that purchasers of the wok do not spend, on average, sufficient time super-heating their woks, he will begin shipping with each wok a comprehensive wok-seasoning guide [redacted] prescribing [redacted] no less than forty minutes super-heating [redacted] during the seasoning process.

(a) Give the null and alternate hypotheses which are of interest to Mr. KG.

Let μ be the mean number of minutes purchasers spend super-heating their woks.

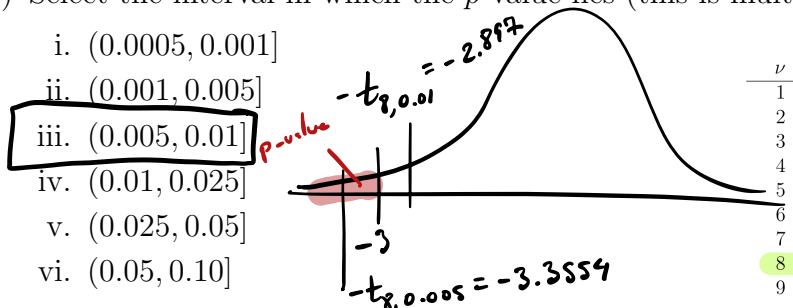
$H_0: \mu \geq 40$ vs $H_1: \mu < 40$

(b) Compute the test statistic of the test for testing the hypotheses in part (a).

$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} = \frac{35 - 40}{5/\sqrt{9}} = -3.$$

(c) Select the interval in which the p-value lies (this is multiple choice):

i. $(0.0005, 0.001]$
ii. $(0.001, 0.005]$
iii. $(0.005, 0.01]$
iv. $(0.01, 0.025]$
v. $(0.025, 0.05]$
vi. $(0.05, 0.10]$



Test stat between these numbers

ν \ α	0.100	0.050	0.025	0.010	0.005	0.001	0.0005
1	3.0777	6.3138	12.7062	31.8205	63.6567	318.3088	636.6192
2	1.8856	2.9200	4.3027	6.9646	9.9248	22.3271	31.5991
3	1.6377	2.3534	3.1824	4.5407	5.8409	10.2145	12.9240
4	1.5332	2.1318	2.7764	3.7469	4.6041	7.1732	8.6103
5	1.4759	2.0150	2.5706	3.3649	4.0321	5.8934	6.8688
6	1.4398	1.9432	2.4469	3.1427	3.7074	5.2076	5.9588
7	1.4149	1.8946	2.3646	2.9980	3.4995	4.7853	5.4079
8	1.3968	1.8595	2.3060	2.8965	3.3554	4.5008	5.0413
9	1.3830	1.8331	2.2622	2.8214	3.2498	4.2968	4.7809

(d) Should Mr. KG begin shipping the wok seasoning guide with each wok? Explain your answer.

Yes, he should; the small p-value (less than 0.01) reflects strong evidence against $H_0: \mu \geq 40$.

(e) What assumption, if any, is implicit in the analysis you have carried out?

The analysis assumes the super-heating times are Normally distributed.

4. A 14" cast-iron wok is also sold by KG's discount store and is marketed as an alternative to the 14" carbon steel wok. To better understand customer sentiment around these products, Mr. KG acquires, for each product, 1-to-5-star ratings from a random sample of 40 purchasers. The frequencies of each rating for each product, along with the mean and standard deviation of the ratings for each product, are tabulated here:

Rating	1	2	3	4	5	mean	std. dev
Carbon steel	2	6	18	12	2	3.150	0.921
Cast-iron	4	14	17	3	2	2.625	0.952

(a) Are the ratings of the sampled purchasers drawn from a Normal distribution? Explain why or why not.

Since each rating is a whole number, 1, 2, 3, 4, or 5, the ratings do not have a Normal distribution. The Normal distribution is a continuous distribution with support on the interval $(-\infty, \infty)$.

(b) To what phenomenon owes the fact that difference in mean ratings will be approximately Normally distributed?

To the phenomenon described by the central limit theorem — that sample means behave more and more like Normally distributed random variables as the sample size is increased, regardless of the shape of the population distribution.

(c) Study the R output below and write an assessment to Mr. KG, based on these data, of the customer sentiments around the 14" cast-iron and carbon steel woks. Give a justification of your assessment.

```
        Two Sample t-test

data:  carbon_steel and cast_iron
t = 2.5059, df = 78, p-value = 0.0143
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 0.1078986 0.9421014
sample estimates:
mean of x mean of y
    3.150     2.625
```

The small p-value (0.0143) for the test of equal mean ratings reflects strong evidence in the data that the means are different. Since the sample mean

rating for the carbon steel wok is greater, on avg than there is strong evidence showing that the carbon steel wok is preferred.

5. Ever thoughtful [redacted] of his customers, Mr. KG wishes to send a recipe with each 14" carbon steel wok he ships. To decide which recipe among three possible recipes he should send, he recruits twelve individuals who are culinarily inclined, but who have as yet never cooked with a carbon steel wok; he assigns each individual randomly to one of the three recipes, such that four individuals are assigned to each recipe. Each individual then cooks the assigned recipe in Mr. KG's kitchen with Mr. KG's very own carbon steel wok under the kind tutelage of that same Mr. KG—and afterwards rates the level of overall reward and gratification experienced on a scale of 1 to 10. The ratings given were

Recipe 1	Recipe 2	Recipe 3
7	3	5
8	6	7
7	5	4
8	5	5

From these data, Mr. KG would like to know if it makes any difference which recipe he sends, and, if possible, which recipe would most please those cooking for the first time with a carbon steel wok.

Here is some R output:

```
    Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value  Pr(>F)
recipe     2 17.167  8.5833  7.3571 0.01278 *
Residuals  9 10.500  1.1667
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
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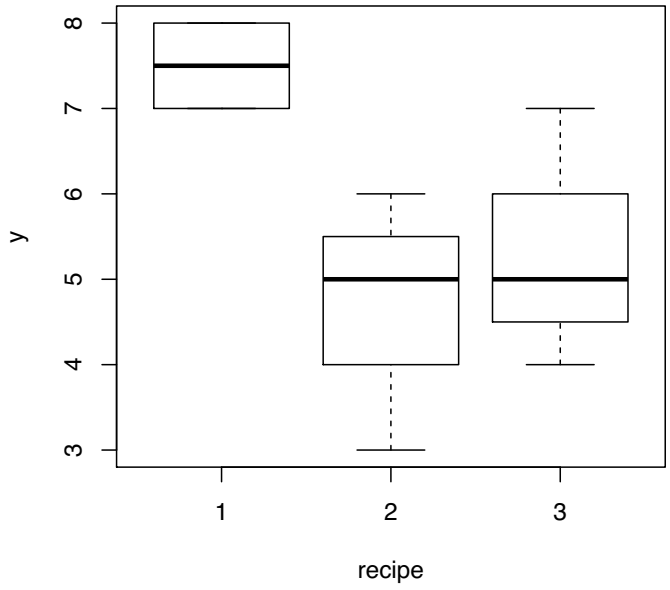
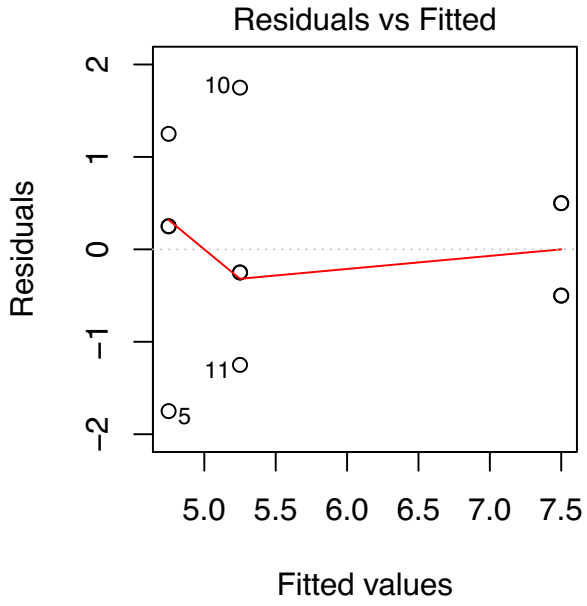
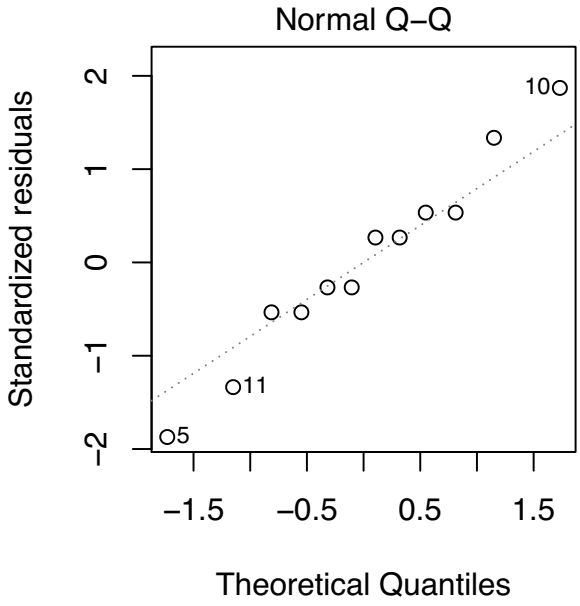






(a) Give the null and alternate hypotheses of interest to Mr. KG.

Let μ_1, μ_2, μ_3 be the average "gratification" levels.

$H_0: \mu_1 = \mu_2 = \mu_3$ vs $H_1: \mu_1, \mu_2, \mu_3$ not all the same.

(b) What does the ratio given by 8.5833 / 1.1667, which appears in the ANOVA table, describe?

This is the F statistic, which is a ratio of between-treatment variation over within-treatment variation.

(c) What is the purpose of the *Residuals vs Fitted* plot?

The purpose is to see whether the variance of the response is the same across treatment groups.

(d) Comment on whether you think the ANOVA assumptions are satisfied.

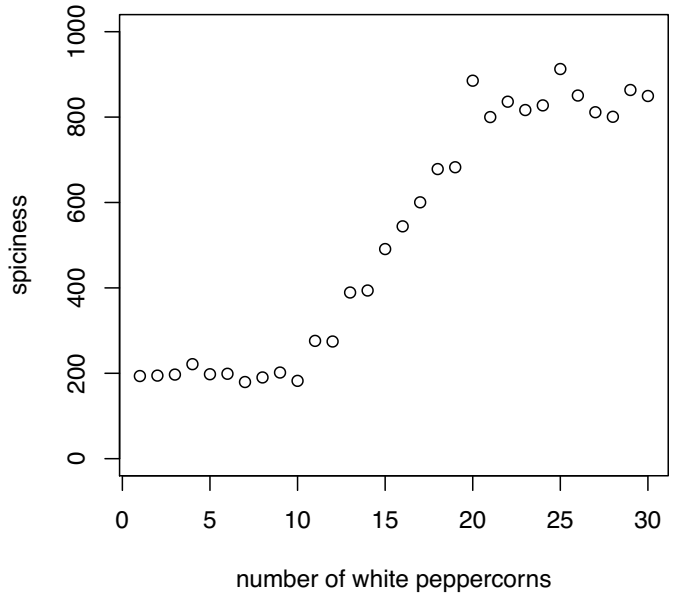
The Normal Q-Q plot indicates some departure from Normality, and the residuals vs fits plot suggests that there may be unequal variances across the treatment groups — but it is hard to tell, since there are only 4 observations in each group. In all, we can probably proceed under the assumptions.

(e) Write a data analysis report for Mr. KG explaining to him what he may conclude from the data. Give justifications for your claims.

There is strong evidence that the means are not all the same, which is reflected in the small p-value (0.01278).

In light of this, Mr. KG should send Recipe 1, which appears (clearly) to have the highest mean.

6. A colleague of Mr. KG complains about the recipe selected on the basis of the study described in the previous question, saying that it is too spicy, owing to the amount of white pepper it calls for. In character with his perfectionism and penchant for statistical rigor, Mr. KG prepares the recipe 30 times, each time with a different number of peppercorns between 1 and 30. The spiciness of the dish is each time rated by his colleague (who does not know how many peppercorns were used) on a scale of 0 to 1000. The study resulted in the data plotted here:



A simple linear regression model is fit to the data. Below is some R output:

```
Call:
lm(formula = Y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max
-165.71  -77.27    6.13   66.35  228.19

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   38.983     36.503   1.068    0.295
x             30.906      2.056  15.031  6.2e-15 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 97.48 on 28 degrees of freedom
Multiple R-squared:  0.8897,Adjusted R-squared:  0.8858
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```
F-statistic: 225.9 on 1 and 28 DF,  p-value: 6.2e-15
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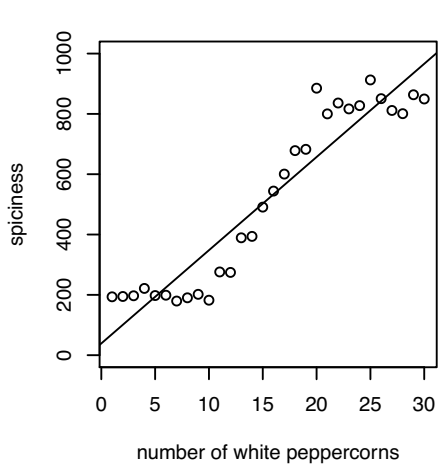
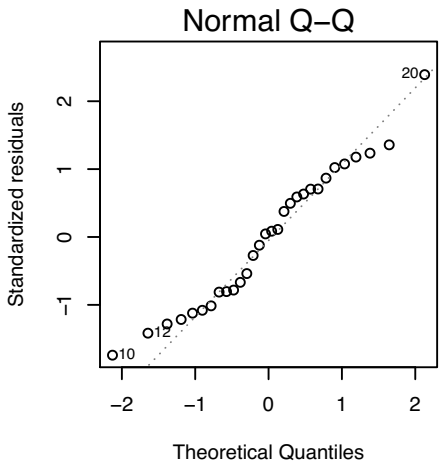
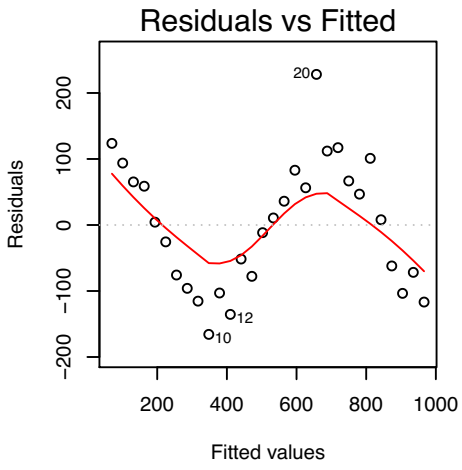






(a) Give the intercept and slope of the least-squares regression line.

$\hat{\beta}_0 = 38.983$

$\hat{\beta}_1 = 30.906$

(b) State whether the assumptions of the linear regression model are satisfied for these data. Explain why they are or are not satisfied.

They are not.
The true relationship does not appear to be linear.
The least squares line does not pass through the data points well — and this is emphasized in the Residuals vs Fits plot.

Mr. KG decides to focus on the relationship between the spiciness level and the number of peppercorns while the latter is between 10 and 20. Ignoring the part of the data with a number of peppercorns outside of this range, the above R output becomes:

```
Call:
lm(formula = Y[10:20] ~ x[10:20])

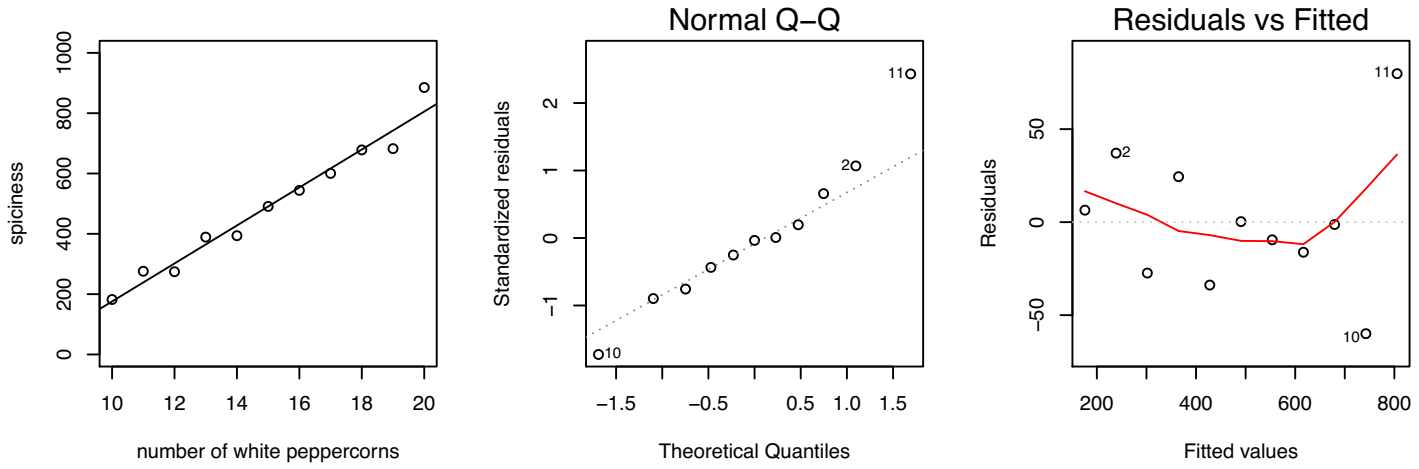
Residuals:
```

```
    Min      1Q  Median      3Q     Max
-59.956 -21.764  -1.273  15.438  79.812

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -453.711     58.109  -7.808 2.69e-05 ***
x[10:20]      62.960      3.791  16.609 4.64e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 39.76 on 9 degrees of freedom
Multiple R-squared:  0.9684,Adjusted R-squared:  0.9649
F-statistic: 275.9 on 1 and 9 DF,  p-value: 4.641e-08
```

slope



(c) In the second analysis, what is the interpretation of the value 62.960 appearing in the R output?

Each additional peppercorn increases the expected spiciness level by 62.690!

(d) In the second analysis the value R^2 is reported as 0.9684. Give a careful interpretation of this value.

This is the proportion of the variability in spiciness level explained by the number of white peppercorns.

7. Mr. KG, interested in whether his customers tend to buy wok spatulas along with their woks, draws a sample of 100 recent orders from KG's Discount Store. Among the 100 orders sampled, 50 orders included a wok and 20 orders included a wok spatula. Of the 50 orders that included a wok, 15 also included a wok spatula.

   (a) Fill out the counts in the table below, including row and column totals, to summarize the 100 sampled orders.

		Wok		
		Yes	No	
Wok spatula	Yes	15	5	20
	No	35	95	80
		50	50	100

   (b) Give the table of expected counts under the null hypothesis of no association.

$\frac{50 \cdot 20}{100}$	$\frac{50 \cdot 20}{100}$
$\frac{50 \cdot 80}{100}$	$\frac{50 \cdot 80}{100}$

$=$

10	10
40	40

   (c) The output of the chi-squared test for association is given below. Based on the output, write for Mr. KG your conclusion concerning an association between buying a wok and buing a wok spatula.

```
Pearson's Chi-squared test

data:  M
X-squared = 6.25, df = 1, p-value = 0.01242
```

Since the p-value is quite small, there is fairly strong evidence to conclude that purchasing a wok and purchasing a wok spatula are not independent events, i.e. there is an association.