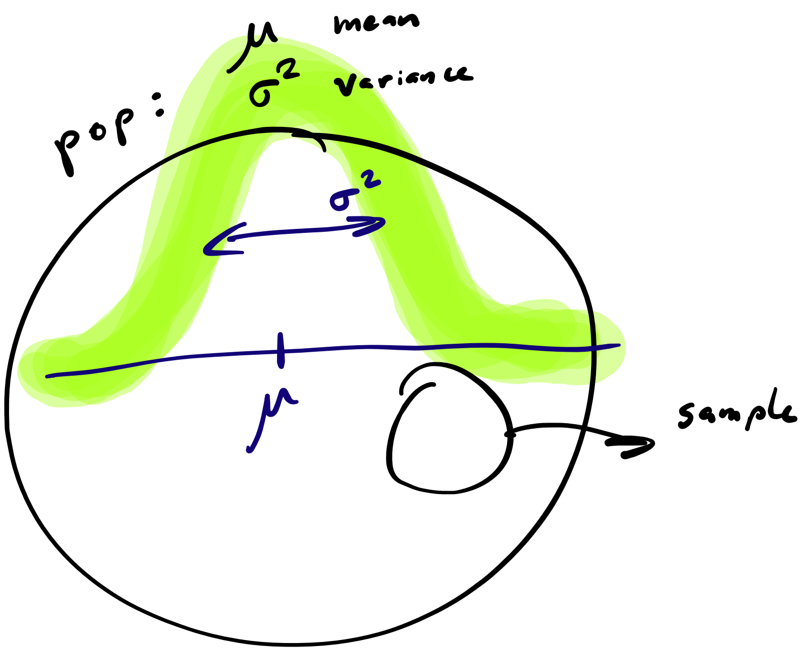
STAT 516 Lec 01

Inference on the mean and variance of a Normal population

Karl Gregory

2024-01-09



$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

Setup

— independent

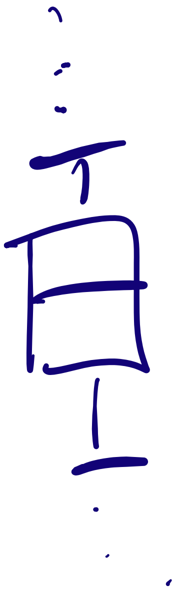
Throughout let $X_1, \ldots, X_n \overset{\text{ind}}{\sim}$ Normal(μ, σ^2).

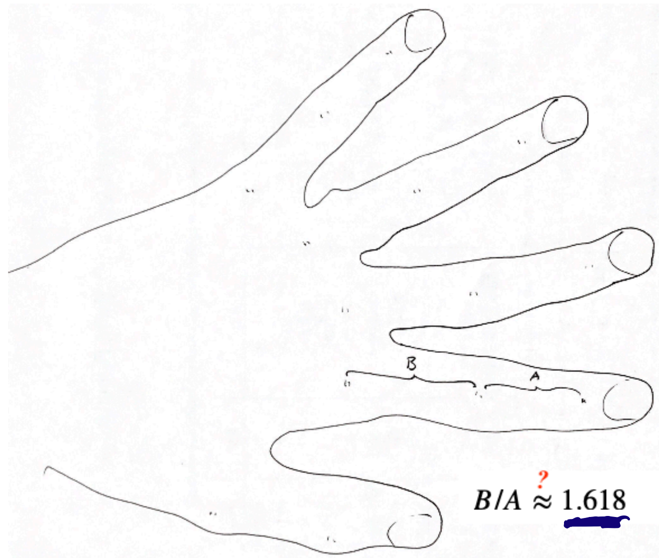
In this lecture we review how to:

1. Estimate μ and σ^2.
2. Build confidence intervals for μ and σ^2.
3. Test hypotheses concerning μ and σ^2.
4. Choose the sample size.

We call $X_1, \ldots, X_n$ a random sample.

Golden ratio example:





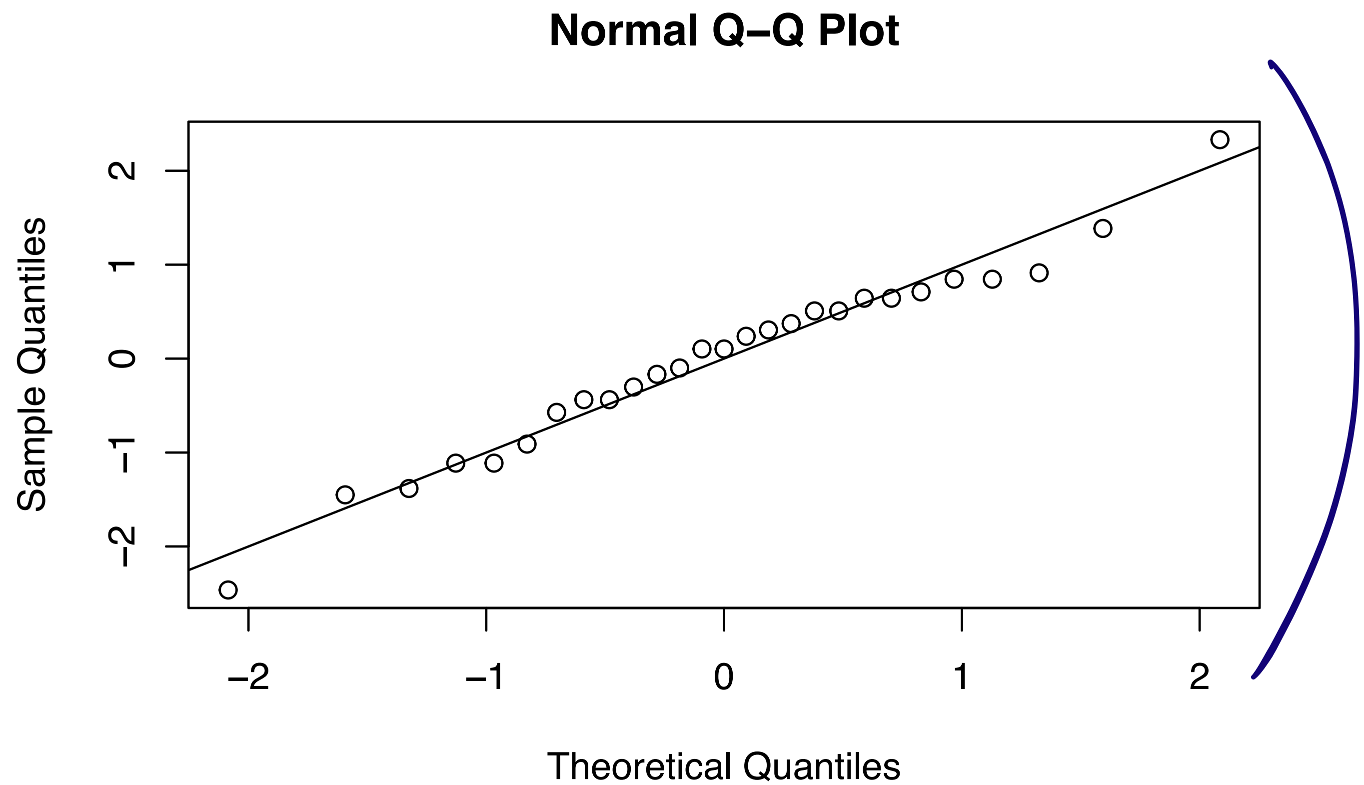
A class of $n = 27$ students measured B/A on themselves:

```r
gr <- c(1.66, 1.61, 1.62, 1.69, 1.58, 1.43, 1.66,
        1.69, 1.58, 1.20, 1.52, 1.60, 1.55, 1.67,
        1.77, 1.50, 1.64, 1.54, 1.40, 1.36, 1.50,
        1.40, 1.35, 1.48, 1.64, 1.91, 1.70)
```

What is the true mean of B/A? Could it be the golden ratio?

Check if B/A measurements come from a Normal distribution.

```r
qqnorm(scale(gr))
abline(0,1)
```



Estimation

Based on $X_1, \ldots, X_n$, define the sample statistics

- $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$
- $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

Then $\bar{X}_n$ and S_n^2 are unbiased estimators of μ and σ^2, respectively.

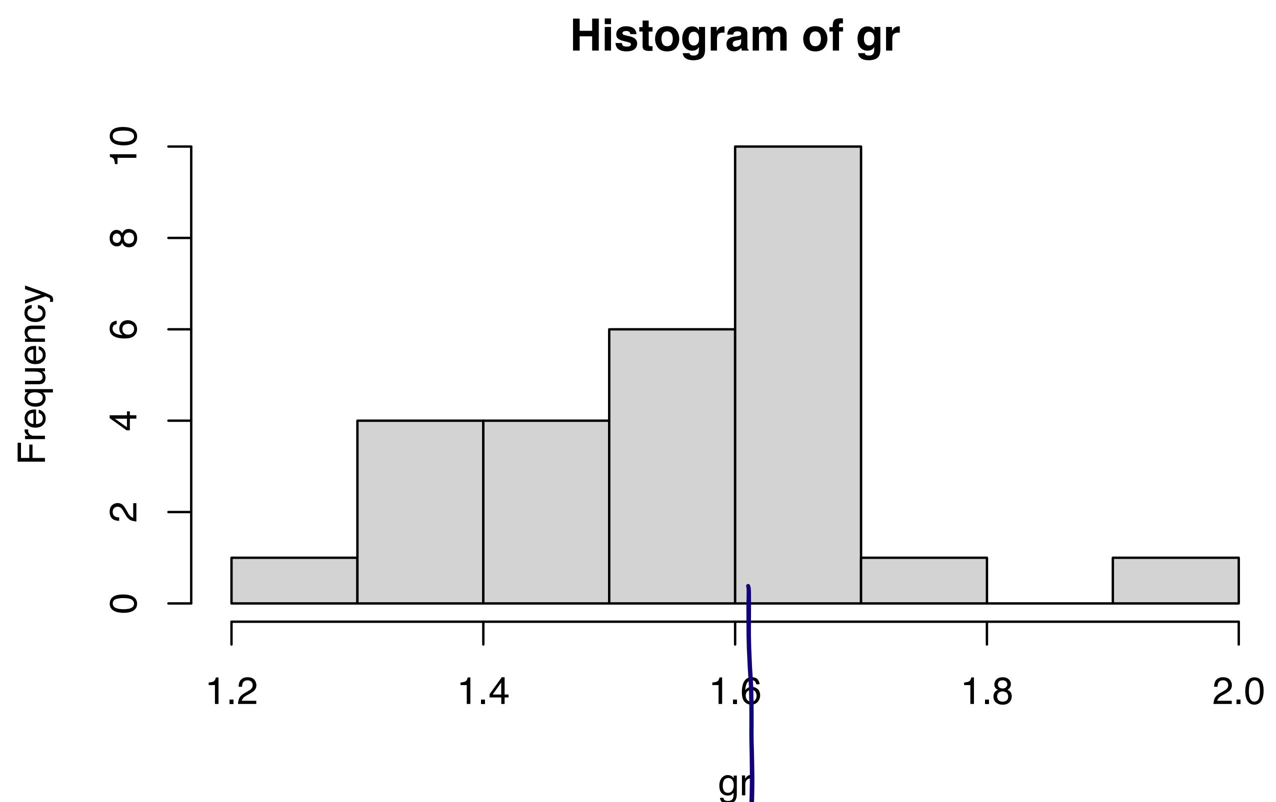
$$\left.\begin{aligned} \mathbb{E}\,\bar{X}_n &= \mu \\ \mathbb{E}\,S_n^2 &= \sigma^2 \end{aligned}\right)$$ Unbiasedness

Golden ratio example (cont):

We have $\bar{X}_n = \texttt{mean(gr)} = 1.565$ and $S_n^2 = \texttt{var(gr)} = 0.0219$.

```
hist(gr)
```

$n=27$



1.618

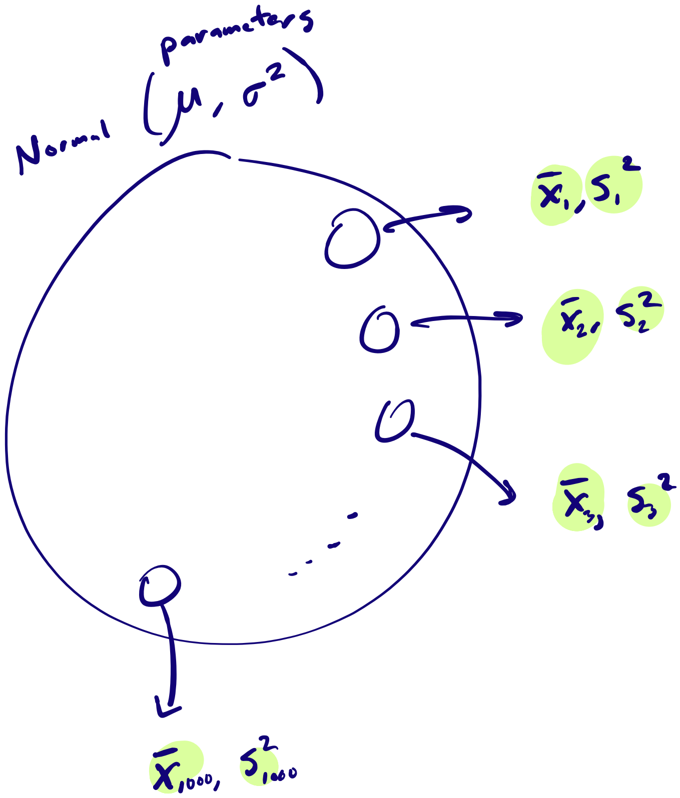
Important sampling distribution results

σ = population std. dev. σ^2 = pop variance

S_n = sample std. dev S_n^2 = sample variance

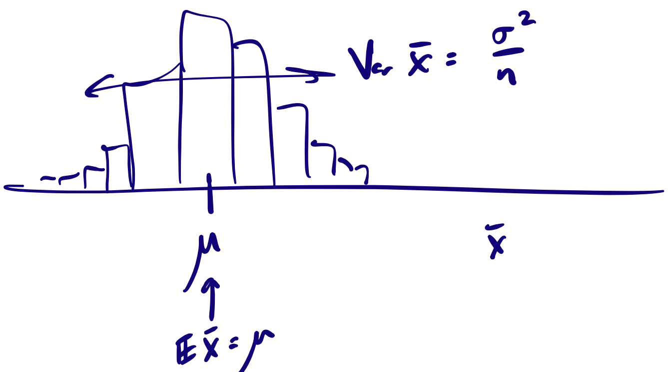
Provided $X_1, \ldots, X_n \overset{\text{ind}}{\sim}$ Normal(μ, σ^2), we have

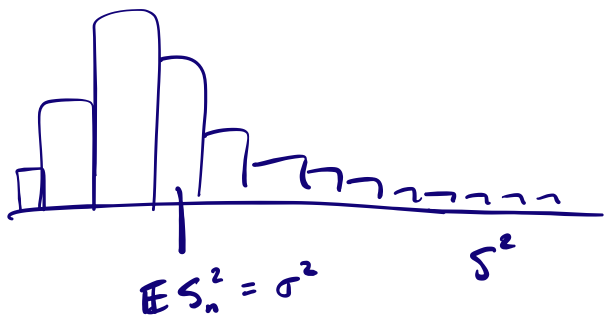
- $\dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim$ Normal$(0, 1)$
- $\dfrac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$
- $\boxed{\dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}}$



Hist. of $\bar{x}_1, \ldots, \bar{x}_{1000}$

$s_1^2, \ldots, s_{1000}^2$





$$\bar{x} \sim \text{Normal}\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0,1) \quad (z\text{-table})$$

Discuss: Anatomy of chi-square and t random variables

- $Z_1, \dots, Z_m \overset{\text{ind}}{\sim} \mathsf{Normal}(0, 1) \implies Z_1^2 + \cdots + Z_m^2 \sim \chi^2_m$.
- $Z \sim \mathsf{Normal(0,1)} \perp\!\!\!\perp W \sim \chi^2_m \implies \dfrac{Z}{\sqrt{W/m}} \sim t_m$.

Relate these to the results on the previous slide.

Simulation illustrating sampling distribution results:

```
sims <- 1000
mu <- 1
sigma <- 1/2
n <- 8
Tn <- numeric(sims)
Wn <- numeric(sims)
for(s in 1:sims){

  X <- rnorm(n,mu,sigma)
  sn <- sd(X)
  xbar <- mean(X)
  Tn[s] <- sqrt(n)*(xbar - mu) / sn
  Wn[s] <- (n-1)*sn^2 / sigma^2

}
```

$1, 2, \ldots, \text{sims}$

$\leftarrow S_n$

$\leftarrow \bar{X}_n$

$= \dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$

$= \dfrac{(n-1)S_n^2}{\sigma^2}$

1000 times:

Draw a sample of size n from Normal(μ, σ^2)

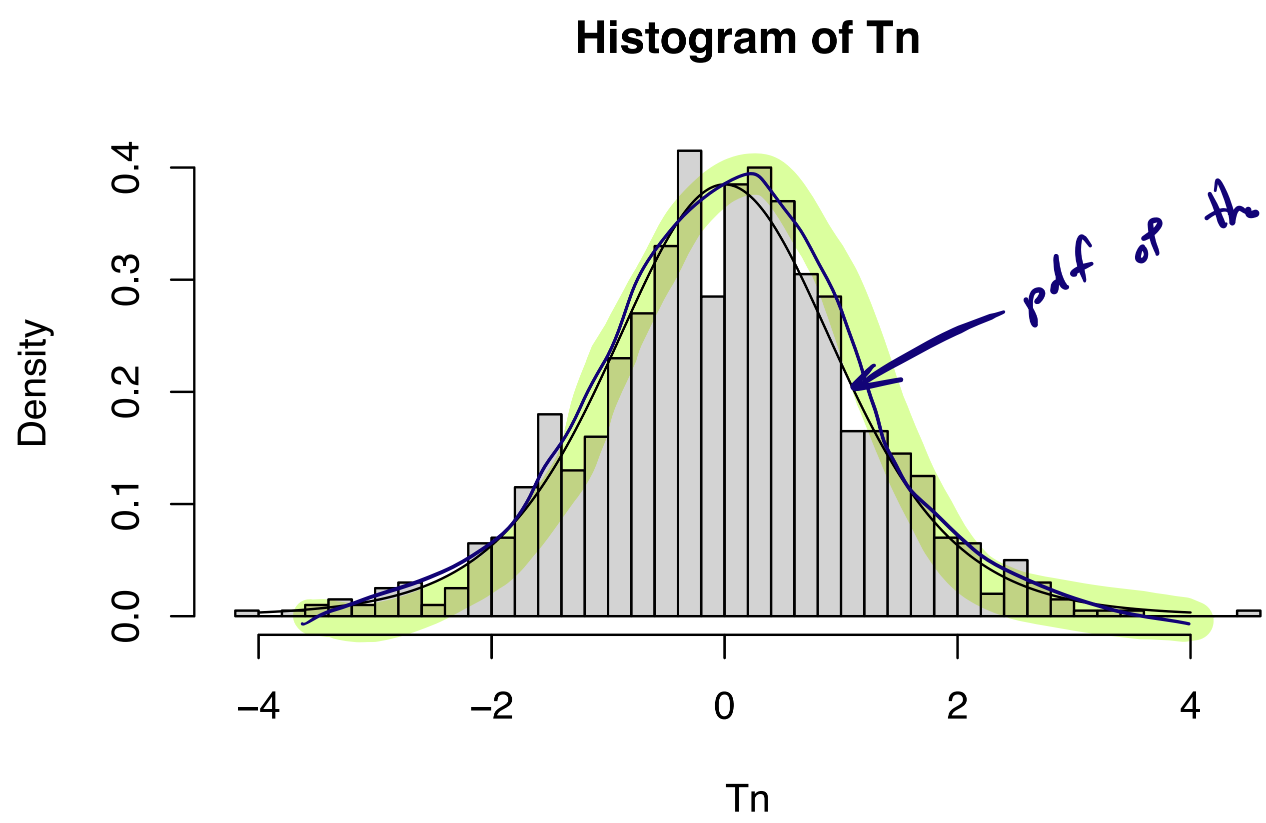
Compute $T = \dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$

$W = \dfrac{(n-1)S_n^2}{\sigma^2}$

Then see if $T_1, \ldots, T_{1000} \sim t_{n-1}$

$W_1, \ldots, W_{1000} \sim \chi^2_{n-1}$

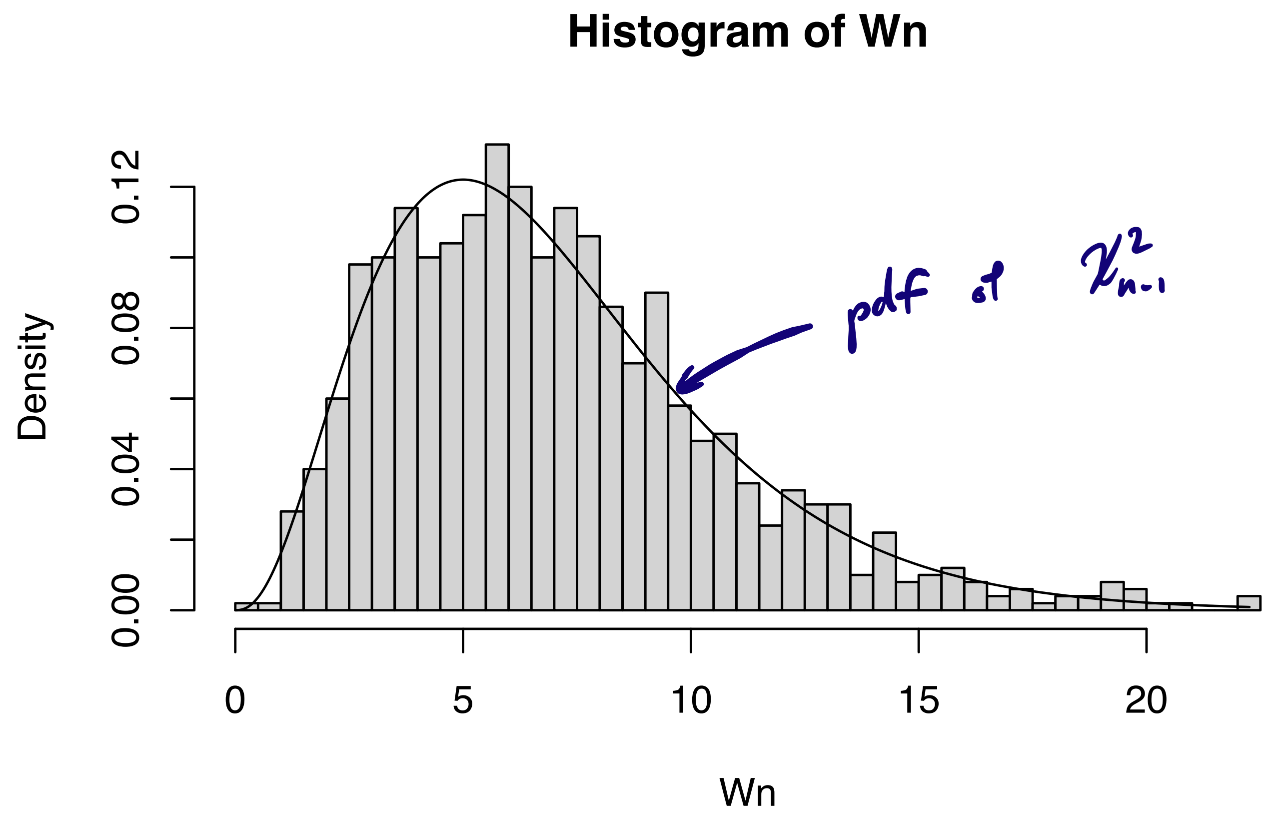
```
hist(Tn,freq = FALSE,breaks = 50)
x <- seq(-4,4,length = 500)
lines(dt(x,n-1)~x)
```



pdf of the t_{n-1} dist.

```r
hist(Wn,freq = FALSE,breaks = 50)
x <- seq(0,max(Wn),length = 500)
lines(dchisq(x,n-1)~x)
```

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$$



Confidence intervals for the mean and variance

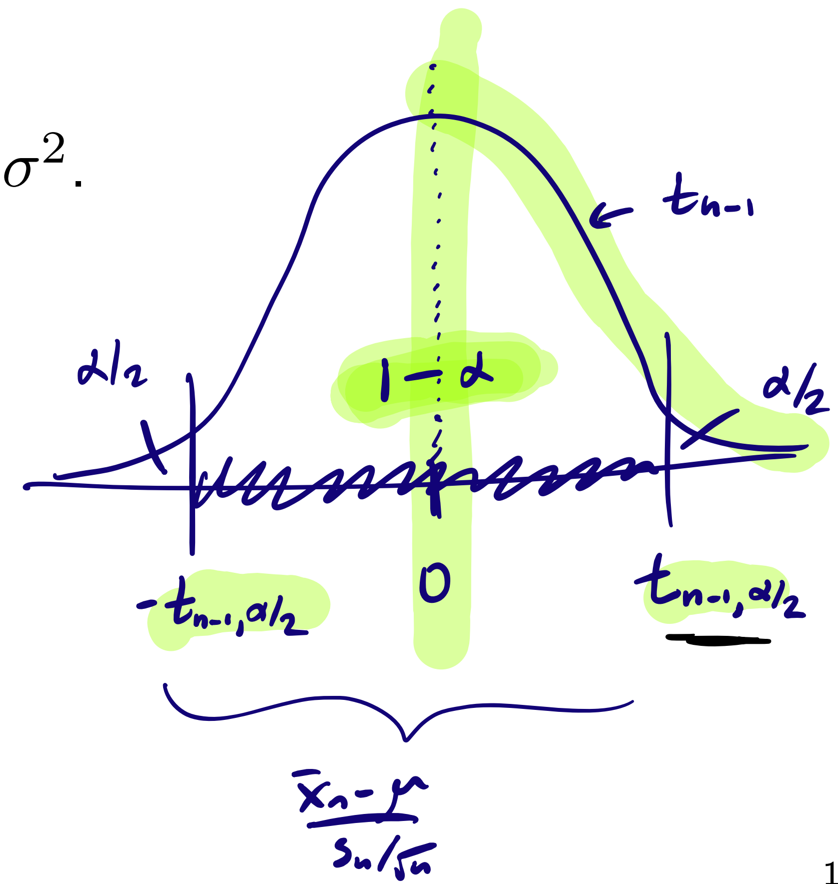
95% C.I. α

The sampling distribution results give $(1-\alpha)100\%$ CIs as

- $\boxed{\bar{X}_n \pm \boxed{t_{n-1,\alpha/2}}\frac{S_n}{\sqrt{n}}}$ for μ.
- $\left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}}\right)$ for σ^2.

Exercise: Derive the above.

If $\frac{\bar{X}_n-\mu}{S_n/\sqrt{n}} \sim t_{n-1}$



$$P\left(-t_{n-1,\alpha/2} < \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} < t_{n-1,\alpha/2}\right) = 1-\alpha$$

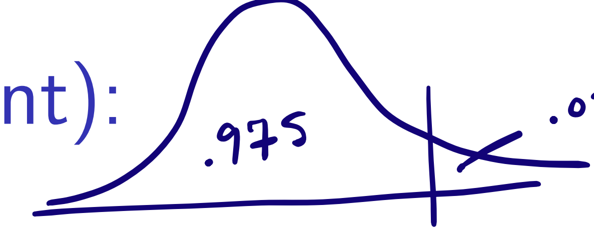
$$\vdots$$

$$P\left(\underbrace{\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}}_{\text{lower}} < \mu < \underbrace{\bar{X}_n + t_{n-1}\frac{S_n}{\sqrt{n}}}_{\text{upper}}\right) = 1-\alpha$$

So an interval which covers μ with prob $1-\alpha$ is

$$\bar{X}_n \pm t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}.$$

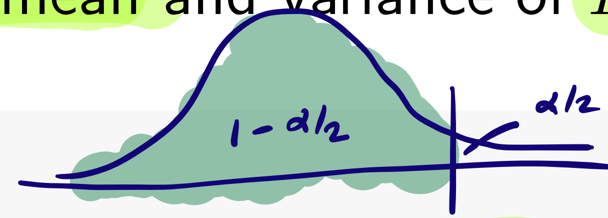
Golden ratio example (cont):



$n=27$

$t_{27-1,\,0.025} = 2.056 = qt(.975, 26)$

Build 95% CIs for population mean and variance of B/A values:



$t_{n-1,\alpha/2}$

$t_{n-1,\alpha/2}$ $\quad \bar{x}_n \quad$ df $\quad S_n$

```
alpha <- 0.05
n <- length(gr)

lomu <- mean(gr) - qt(1-alpha/2,n-1) * sd(gr)/sqrt(n)
upmu <- mean(gr) + qt(1-alpha/2,n-1) * sd(gr)/sqrt(n)

losgs <- (n-1) * var(gr) / qchisq(1-alpha/2,n-1)
upsgs <- (n-1) * var(gr) / qchisq(alpha/2,n-1)
```

The 95% CI for μ is (1.506,1.623). For σ^2 it is (0.014,0.041).

Is $\mu = 1.618$?

Statistical inference

Confidence intervals

Hypothesis testing

Testing hypotheses about the mean

Making α smaller means you require stronger evidence to reject H_0.

Consider testing hypotheses about μ of the form

$$H_0\colon \mu \geq \mu_0 \quad \text{or} \quad H_0\colon \mu = \mu_0 \quad \text{or} \quad H_0\colon \mu \leq \mu_0$$
$$H_1\colon \mu < \mu_0 \qquad\qquad H_1\colon \mu \neq \mu_0 \qquad\qquad H_1\colon \mu > \mu_0.$$

Reject or fail to reject H_0 based on the value of the test statistic

$$\boxed{T_{\text{stat}} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}.} = \text{\# standard devs that } \bar{x} \text{ is away from } \mu_0$$

Rejection rules for the above at significance level α are

$$T_{\text{stat}} < -t_{n-1,\alpha} \quad \text{or} \quad |T_{\text{stat}}| > t_{n-1,\alpha/2} \quad \text{or} \quad T_{\text{stat}} > t_{n-1,\alpha}.$$

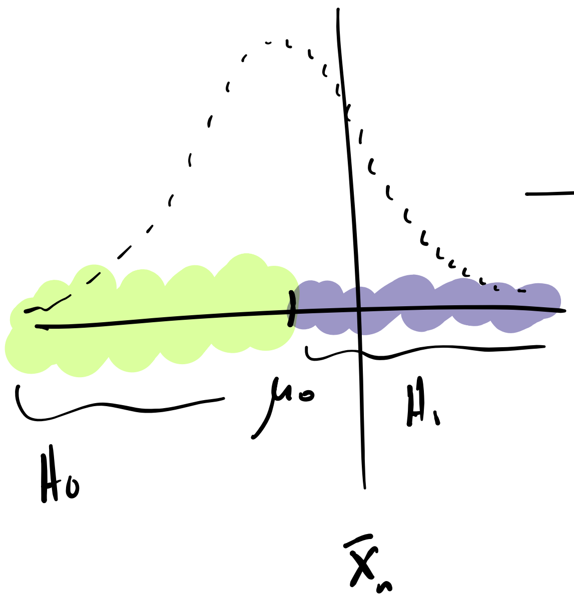
The corresponding p-values are, with $T \sim t_{n-1}$, the probabilities

$$P(T < T_{\text{stat}}) \quad \text{or} \quad 2 \times P(T > |T_{\text{stat}}|) \quad \text{or} \quad P(T > T_{\text{stat}}).$$

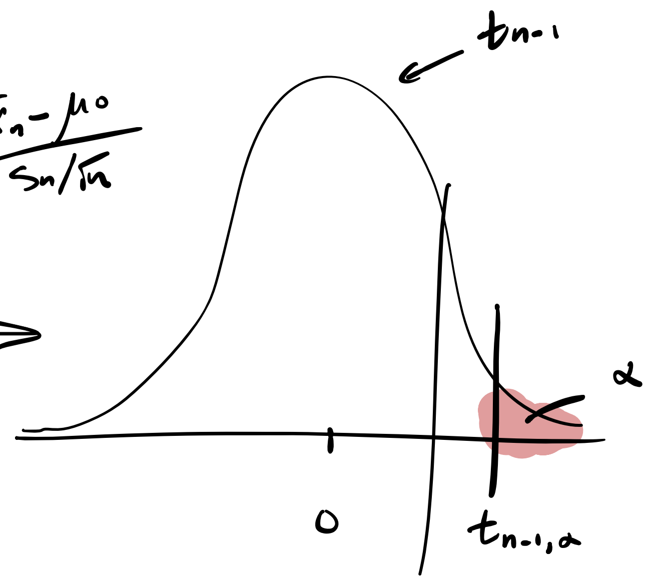
p-value: the smallest significance level at which you would still reject H_0.

$H_0: \mu \leq \mu_0$

$H_1: \mu > \mu_0$



$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$



$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

Keep Prob of Type I error $\leq \alpha$.

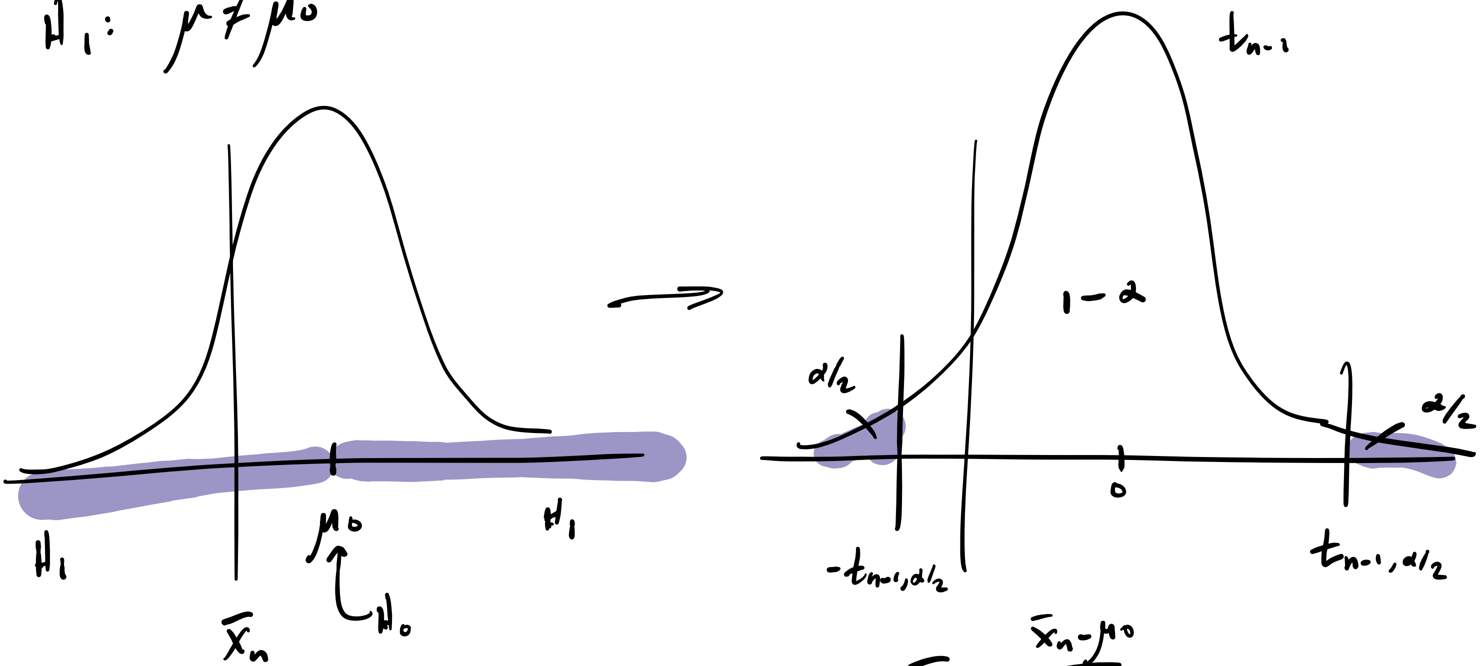
Reject H_0 when $T_{test} > t_{n-1,\alpha}$.

	H_0 true	H_0 false
Rej H_0	Type I	Correct
FTR H_0	Correct	Type II

Outcomes of statistical inference

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$



$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

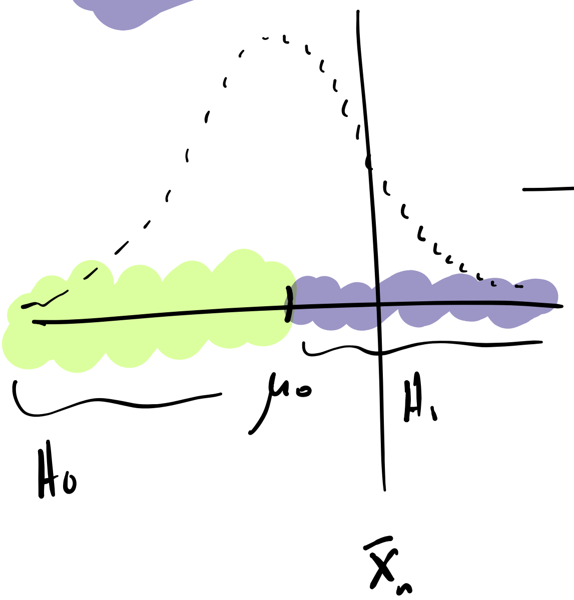
Reject H_0 if $T_{test} < -t_{n-1,\alpha/2}$

or $T_{test} > t_{n-1,\alpha/2}$.

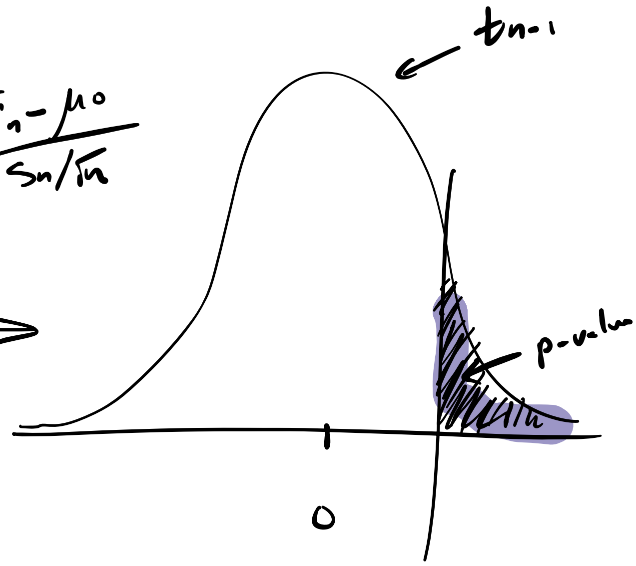
Same as reject H_0 if $|T_{test}| > t_{n-1,\alpha/2}$

$H_0: \mu \le \mu_0$

$H_1: \mu > \mu_0$



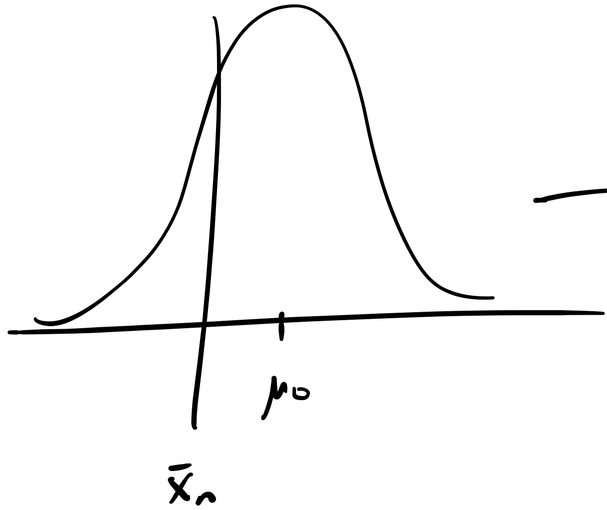
$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

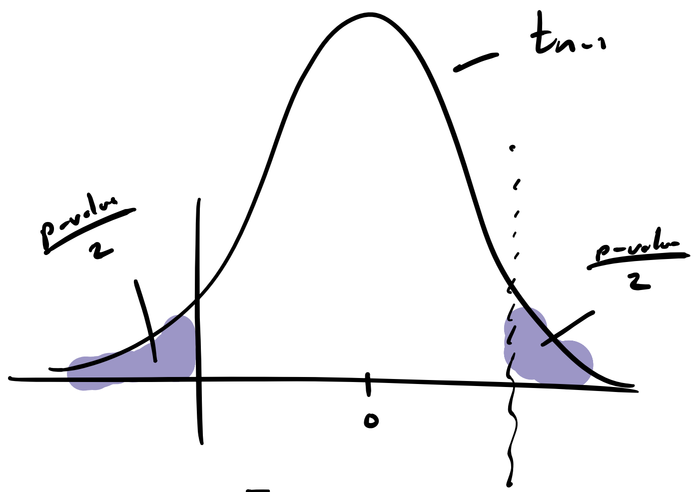


$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

$H_0: \mu = \mu_0$

$H_1: \mu \ne \mu_0$

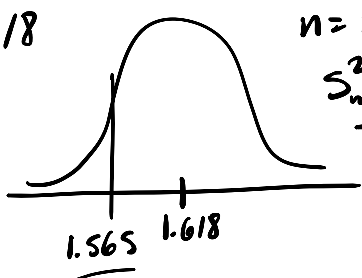




$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

$H_0: \mu = 1.618$

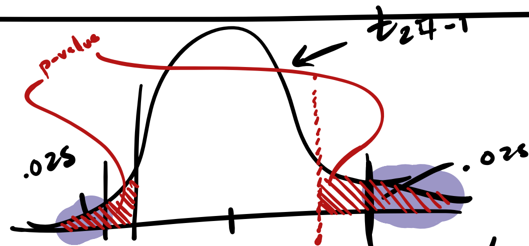
$H_1: \mu \ne 1.618$



$\bar{X}_n = 1.565$

$n = 27$

$S_n^2 = 0.0219$



$$-1.866 = \frac{1.565 - 1.618}{\sqrt{0.0219}/\sqrt{27}}$$

$$2.056 = t_{27-1,\,0.025} = qt(.975,\ 27-1)$$

Golden ratio example (cont):

Test H_0: $\mu = \underbrace{1.618}_{\mu_0}$ vs H_1: $\mu \neq 1.618$ at $\alpha = 0.05$ based on data.

$T_{stat} = \dfrac{\bar{x}_n - 1.618}{s_n/\sqrt{n}}$

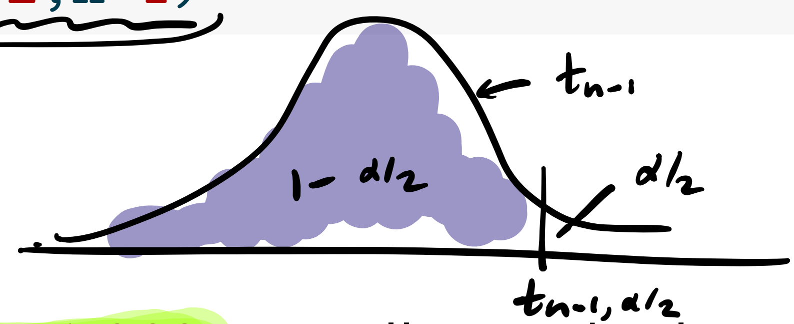
$\bar{x}_n$ μ_0

```
alpha <- 0.05
Tstat <- (mean(gr) - 1.618) / (sd(gr) / sqrt(n))
abs(Tstat) > qt(1-alpha/2,n-1)
```

s_n $\sqrt{n}$

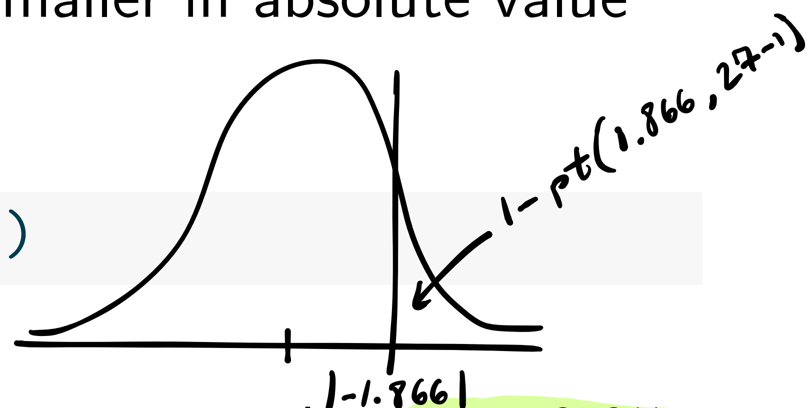
$t_{n-1,\alpha/2}$

```
[1] FALSE
```



Fail to reject H_0 since T_{stat} = -1.866 is smaller in absolute value than $t_{n-1,\alpha/2} = 2.056$.

```
pval <- 2*(1 - pt(abs(Tstat),n-1))
```



Equivalently, the p-value, which is 0.073, is greater than $\alpha = 0.05$.

Fail to reject at $\alpha = 0.05$.

The `t.test()` function in R

The function `t.test()` tests H_0: $\mu = 0$ vs H_1: $\mu \neq 0$ by default.

```
t.test(gr)
```

data

$T_{test} = \frac{\bar{x}_n - 0}{s_n/\sqrt{n}}$

$n=27$, $df = n-1 = 27-1 = 26$

```
	One Sample t-test

data:  gr
t = 54.902, df = 26, p-value < 2.2e-16
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 1.506228 1.623401
sample estimates:
mean of x 
 1.564815 
```

before (1.506,1.623).

$\bar{x}_n$

Now test H_0: $\mu = 1.618$ versus H_1: $\mu \neq 1.618$, ask for 99% CI.

```
t.test(gr, mu = 1.618, conf.level = 0.99)
```

```
	One Sample t-test

data:  gr
t = -1.866, df = 26, p-value = 0.07336
alternative hypothesis: true mean is not equal to 1.618
99 percent confidence interval:
 1.485616 1.644013
sample estimates:
mean of x
 1.564815
```

$T_{test} = \frac{\bar{X}_n - 1.618}{S_n/\sqrt{n}}$

n-1

Now test H_0: $\mu \leq 1.618$ versus H_1: $\mu > 1.618$.

```
t.test(gr, mu = 1.618, alternative = "greater")
```

```

	One Sample t-test

data:  gr
t = -1.866, df = 26, p-value = 0.9633
alternative hypothesis: true mean is greater than 1.618
95 percent confidence interval:
 1.516202      Inf
sample estimates:
mean of x
 1.564815
```

Testing hypotheses about the variance

Consider testing hypotheses about σ^2 of the form

$$\begin{array}{l} H_0\colon \sigma^2 \geq \sigma_0^2 \\ H_1\colon \sigma^2 < \sigma_0^2 \end{array} \quad \text{or} \quad \begin{array}{l} H_0\colon \sigma^2 \leq \sigma_0^2 \\ H_1\colon \sigma^2 > \sigma_0^2 \end{array}$$

Reject or fail to reject H_0 based on the value of the test statistic

$$W_{\text{stat}} = \frac{(n-1)S_n^2}{\sigma_0^2}.$$

Rejection rules for the above at significance level α are

$$W_{\text{stat}} < \chi^2_{n-1,1-\alpha} \quad \text{or} \quad W_{\text{stat}} > \chi^2_{n-1,\alpha}$$

The corresponding p-values are, with $W \sim \chi^2_{n-1}$, the probabilities

$$P(W < W_{\text{stat}}) \quad \text{or} \quad P(W > W_{\text{stat}}).$$

Golden ratio example (cont):

Test H_0: $\sigma^2 \geq 0.03$ vs H_1: $\sigma^2 < 0.03$ at $\alpha = 0.05$ based on data.

```
alpha <- 0.05
Wstat <- (n-1)*var(gr) / 0.03
Wstat < qchisq(alpha,n-1)
```

```
[1] FALSE
```

FTR H_0 since $W_{\text{stat}} = 19.009$ is not less than $\chi^2_{n-1,1-\alpha} = 15.379$.

```
pval <- pchisq(Wstat,n-1)
```

Equivalently, the p-value, which is 0.164, is greater than $\alpha = 0.05$.

Sample size calculations

$$\bar{X}_n \pm t_{n-1,\alpha/2} \frac{S_n}{\sqrt{n}}$$

or $$\bar{X}_n \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$ (σ known)

We can choose a sample size based on the desired:

a. Width of a confidence interval.

b. Power of a test to reject H_0 when it is false.

	H_0 true	H_0 false	
Rej H_0	Type I error	Correct	γ^* — we want this to have high probability
FTR H_0	Correct	Type II error	β^*

Sample size required to achieve desired CI width

$$\underbrace{\bar{X}_n \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{\text{Easier to work with.}}$$

A CI for μ takes the form $\bar{X}_n \pm M$, where

- $M = \boxed{z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$ if σ is known
- $M = \boxed{t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}}$ if σ is unknown

For ease, use the "σ-known" version.

If one wants $M \leq M^*$, find smallest n such that $z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq M^*$.

$$\Leftrightarrow \frac{z_{\alpha/2}\,\sigma}{M^*} \leq \sqrt{n}$$

So take $n = \left\lceil \left(\frac{z_{\alpha/2}\sigma}{M^*}\right)^2 \right\rceil$, where $\lceil\cdot\rceil$ rounds up.

$$\Leftrightarrow \left(\frac{z_{\alpha/2}\,\sigma}{M^*}\right)^2 \leq n$$

Must put in a guess for σ.

Golden ratio example (cont):

before $(1.506, 1.623)$. ← 95% based on $n = 27$.

$\approx .12$

$S_n^2 = 0.0219$

$z_{\alpha/2} = z_{0.025} = 1.96 = \text{qnorm}(0.975)$

Find n required to make the 95% CI for μ no wider than 0.08.

$\bar{x}_n \pm M$ $\quad M^* = 0.04$

```
alpha <- 0.05
M <- 0.08/2
sigma_guess <- sd(gr)
nr <- ceiling((qnorm(1-alpha/2) * sigma_guess / M)^2)
nr
```

```
[1] 53
```

$$\left\lceil \frac{(1.96)^2 (0.0219)}{(0.04)^2} \right\rceil = 53.$$

Sample size required to achieve desired power

The power of a test is the probability with which it rejects H_0.

For tests of H_0 concerning the mean μ we write the power as

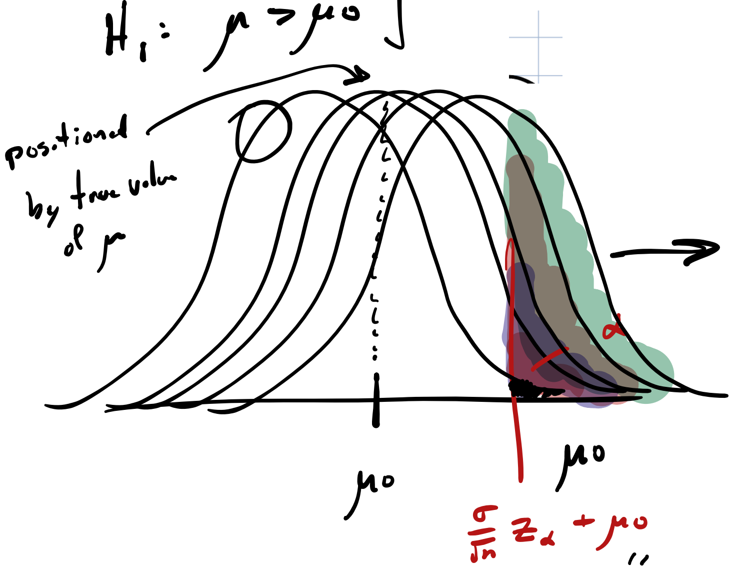
$$\gamma(\mu) = P(\text{Reject } H_0 \text{ when true mean is } \mu) = P_\mu(\text{Reject } H_0).$$

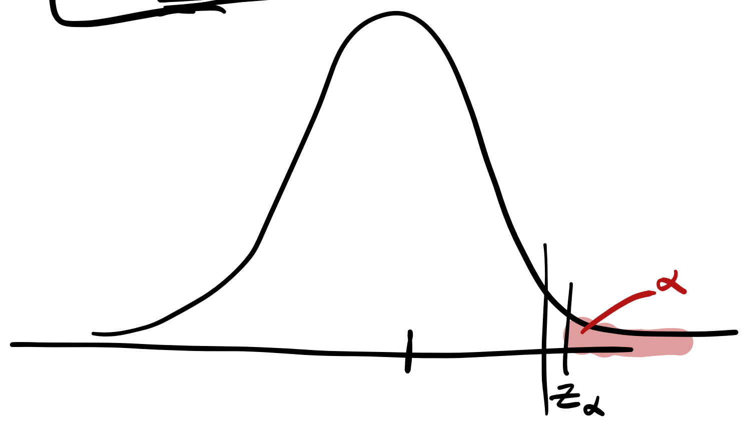
So the power depends on the true value of μ, i.e. is a function of μ.

$H_0: \mu \le \mu_0$

$H_1: \mu > \mu_0$

Rej H_0 if $\boxed{Z_{test} = \dfrac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > z_\alpha}$





$$Z_{test} = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}$$

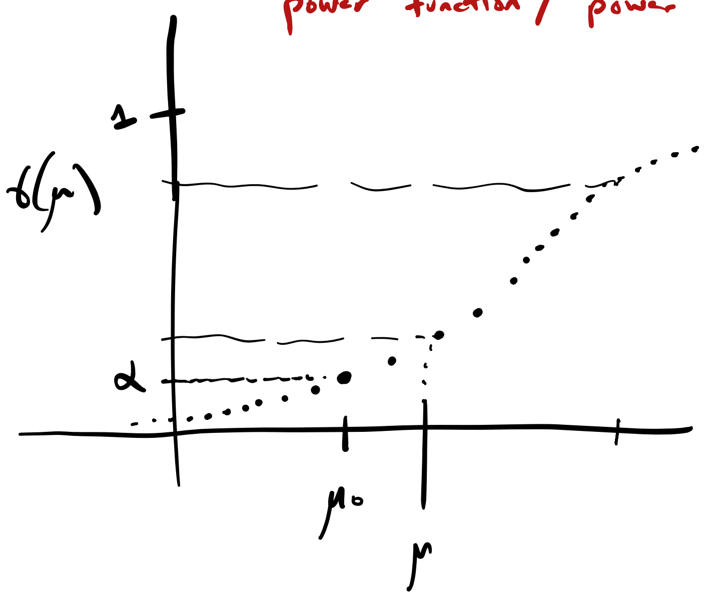
"gamma"

$$P_\mu(\text{Reject } H_0) = \underbrace{\gamma(\mu)}_{\text{power function / power curve}} = P_\mu(Z_{test} > z_\alpha)$$

$$= P_\mu\left(\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > z_\alpha\right)$$

$$= P_\mu\left(\frac{\bar{X}_n - \mu + \mu - \mu_0}{\sigma/\sqrt{n}} > z_\alpha\right)$$

$$= P_\mu\left(\underbrace{\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}_{Z \sim N(0,1)} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} > z_\alpha\right)$$



$$= P\left(Z + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} > z_\alpha\right)$$

$$= P\left(Z > z_\alpha - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)$$

$$= 1 - P\left(Z \le \underbrace{z_\alpha - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}}\right)$$

$$= 1 - \text{pnorm}\left(z_\alpha - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)$$

Exercise: Derive the power functions for the tests of

$$\begin{array}{lll} H_0\colon \mu \geq \mu_0 & \text{and} \quad H_0\colon \mu = \mu_0 & \text{and} \quad H_0\colon \mu \leq \mu_0 \\ H_1\colon \mu < \mu_0 & \phantom{\text{and}} \quad H_1\colon \mu \neq \mu_0 & \phantom{\text{and}} \quad H_1\colon \mu > \mu_0 \end{array}$$

with the rejection rules

$$Z_{\text{stat}} < -z_\alpha \quad \text{and} \quad |Z_{\text{stat}}| > z_{\alpha/2} \quad \text{and} \quad Z_{\text{stat}} > z_\alpha,$$

respectively, where $Z_{\text{stat}} = \dfrac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}$ (σ-known case).

Plot of power curves for right-, left-, and two-sided tests

$$\gamma(\mu) = 1 - \text{pnorm}\left(\underbrace{z_\alpha}_{} - \underbrace{\frac{\mu - \mu_0}{\sigma/\sqrt{n}}}_{}\right)$$

```
alpha <- 0.05
sigma <- 1
n <- 5
mu0 <- 0
mu <- seq(-2,2,length=500)
za <- qnorm(1-alpha)
za2 <- qnorm(1-alpha/2)
d <- sqrt(n) * (mu - mu0) / sigma
rp <- 1 - pnorm(za - d)
lp <- pnorm(-za - d)
rp2 <- 1 - pnorm(za2 - d)
lp2 <- pnorm(-za2 - d)
tsp <- lp2 + rp2
```

$\mu_0 = 0$

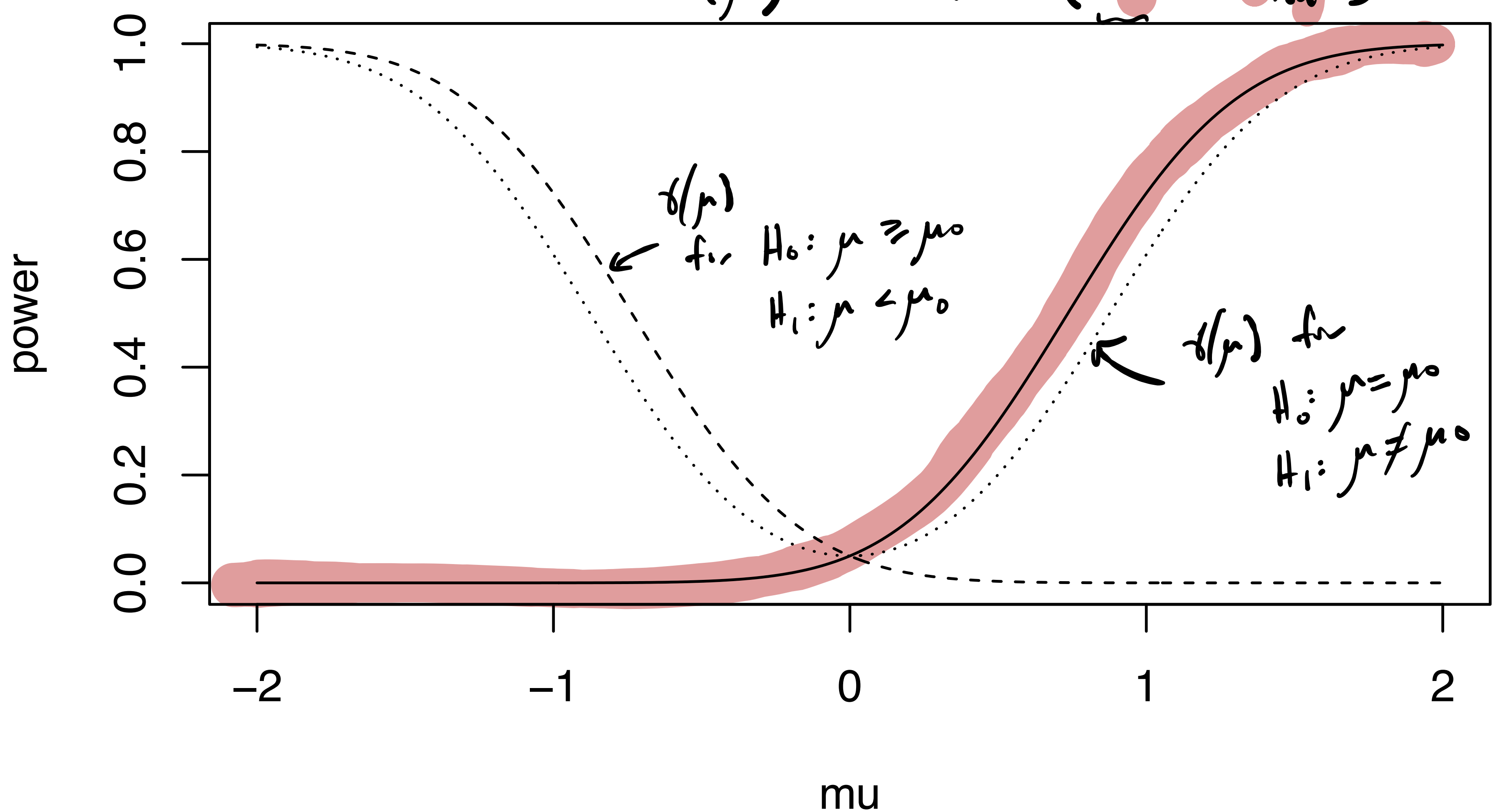
$\mu = -2, \ldots, 2 \ldots$

z_α

$\frac{\mu - \mu_0}{\sigma/\sqrt{n}}$

```
plot(rp ~ mu, type = "l", ylab = "power", xlab = "mu")
lines(lp ~ mu, lty =2)
lines(tsp ~ mu, lty = 3)
```

$\gamma(\mu) = 1 - \text{pnorm}\left(z_\alpha - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)$



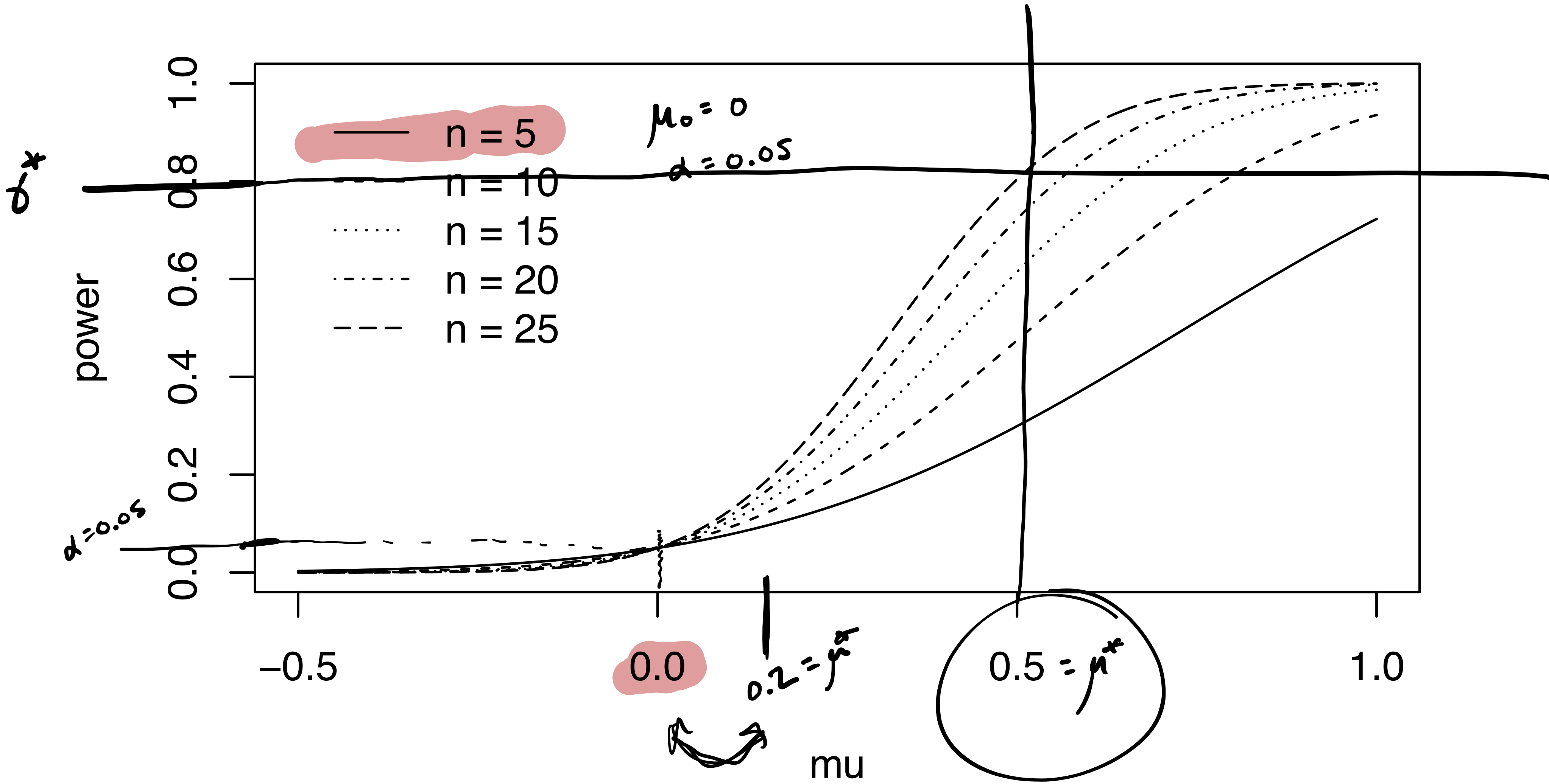
Power curve for right-sided test at various sample sizes

$\gamma(\mu) = 1 - \text{pnorm}\left(z_\alpha - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)$

```
alpha <- 0.05
sigma <- 1
nn <- c(5,10,15,20,25)
mu0 <- 0
mu <- seq(-1/2,1,length=500)
za <- qnorm(1-alpha)
rp <- matrix(NA,500,length(nn))
for(j in 1:length(nn)){
  d <- sqrt(nn[j]) * (mu - mu0) / sigma
  rp[,j] <- 1 - pnorm(za - d)
}
```

n

```r
plot(NA,xlim = range(mu), ylim = c(0,1),
     ylab = "power", xlab = "mu")
for(j in 1:length(nn)) lines(rp[,j] ~ mu, lty = j)
legend(x = min(mu), y = 1,legend = paste("n =",nn),
       lty = 1:length(nn),bty = "n")
```



To choose n based on desired power:

a true value of μ, different from μ_0 by an amount scientifically interesting.

1. Fix μ^* and a desired power γ^*.
2. Find the smallest n guaranteeing power $\geq \gamma^*$ at μ^*.

Example: The test of $H_0\colon \mu \leq \mu_0$ vs $H_1\colon \mu > \mu_0$ with rejection rule $Z_{\text{stat}} > z_\alpha$ has power given by

$$= 1 - P\left(Z \leq z_\alpha - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)$$

$$\gamma(\mu) = 1 - \Phi\left(z_\alpha - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right).$$

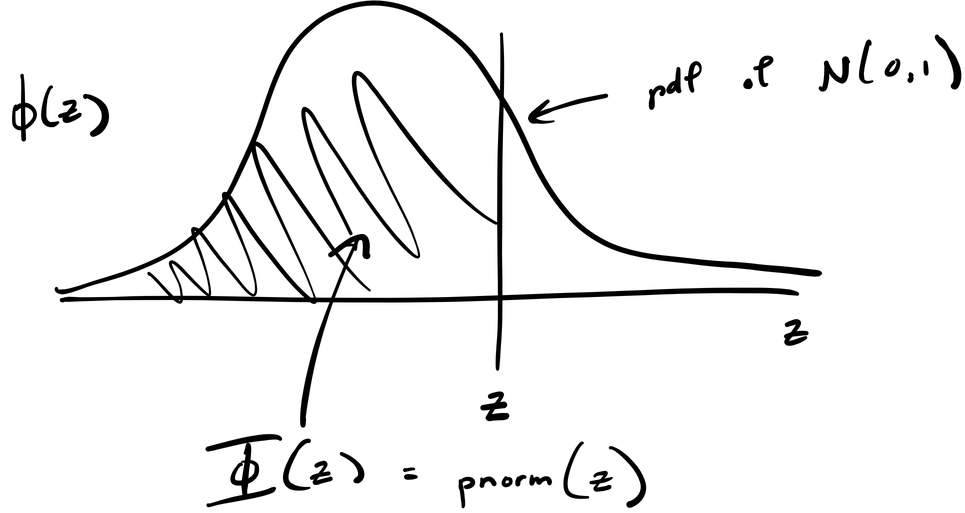
Capital Phi, denote the c.d.f. of the $N(0,1)$.

Fix μ^*, γ^*, find smallest n such that $\boxed{1 - \Phi\left(z_\alpha - \frac{\mu^* - \mu_0}{\sigma/\sqrt{n}}\right)} \geq \gamma^*$.

This gives $n = \left\lceil \frac{\sigma^2(z_\alpha + z_{\beta^*})^2}{(\mu^* - \mu_0)^2} \right\rceil$, where $\beta^* = 1 - \gamma^*$.

γ^* = desired power

β^* = Prob of Type II error



Find smallest n such that

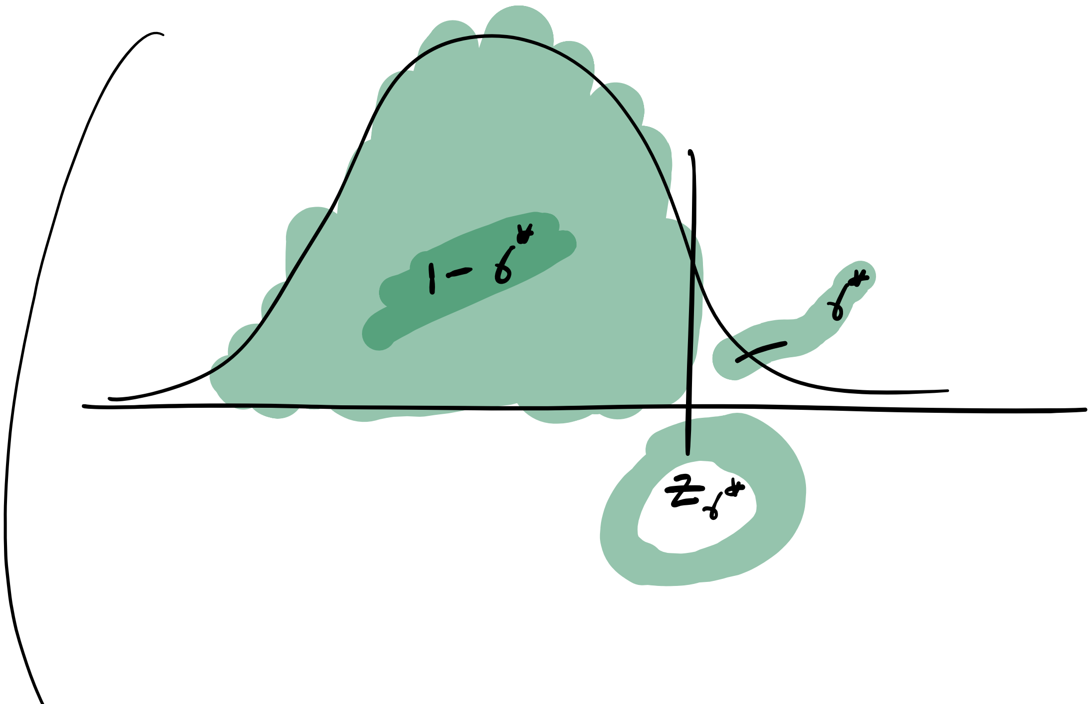
$$P_{\mu^*}(\text{Rej } H_0) \geq \gamma^*$$

$\iff$

$$1 - \Phi\left(z_\alpha - \frac{\mu^* - \mu_0}{\sigma/\sqrt{n}}\right) \geq \gamma^*$$

$\iff$

$$1 - \gamma^* \geq \Phi\left(z_\alpha - \frac{\mu^* - \mu_0}{\sigma/\sqrt{n}}\right)$$



$$\iff \quad z_\alpha - \frac{\mu^* - \mu_0}{\sigma/\sqrt{n}} \le z_{\gamma^*}$$

$$\iff \quad z_\alpha - z_{\gamma^*} \le \frac{\mu^* - \mu_0}{\sigma/\sqrt{n}} = \sqrt{n}\left(\frac{\mu^* - \mu_0}{\sigma}\right)$$

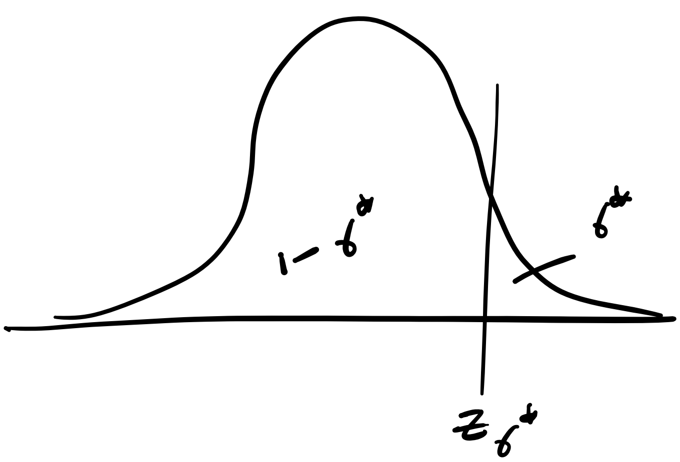
$$\iff \quad \frac{\sigma\left(z_\alpha - z_{\gamma^*}\right)}{\mu^* - \mu_0} \le \sqrt{n}$$

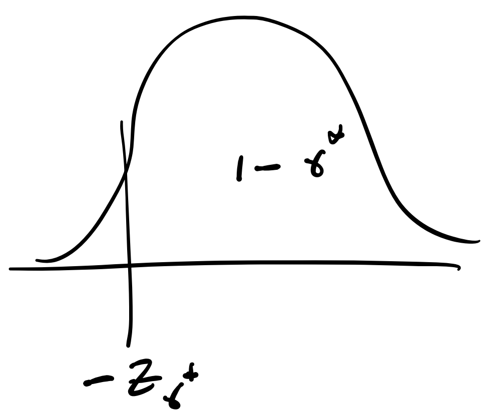
$$\iff \quad \frac{\sigma^2\left(z_\alpha - z_{\gamma^*}\right)^2}{\left(\mu^* - \mu_0\right)^2} \le n.$$

$$\|$$

$$\frac{\sigma^2\left(z_\alpha + z_{\beta^*}\right)^2}{\left(\mu^* - \mu_0\right)^2} \le n$$

$\beta^* = 1 - \gamma^*$





$\|$

z_{β^*} , $\beta^* = 1 - \gamma^*.$

Golden ratio example (cont):

Suppose the true mean of B/A in the population is 1.7.

Give the sample size n required to reject H_0: $\mu \leq 1.618$ vs H_1: $\mu > 1.618$ with power ≥ 0.80. Use $S_n = 0.148$ as a guess of σ.

```
alpha <- 0.05
gm <- 0.80
sigma <- sd(gr)
mu <- 1.7
mu0 <- 1.618
za <- qnorm(1 - alpha)
zb <- qnorm(gm)
nr <- ceiling(sigma^2 * (za + zb)^2 / (mu - mu0)^2)
nr
```

gm $= \gamma^*$, sigma $= S_n$, mu $= \mu^*$, za $= z_\alpha$, zb $= z_{\beta^*}$

$H_0: \mu \leq 1.618$

$H_1: \mu > 1.618$

Suppose, in truth, $\mu = 1.7$.

```
[1] 21
```

$= \left\lceil \frac{\sigma^2 (z_\alpha + z_{\beta^*})^2}{(\mu^* - \mu_0)^2} \right\rceil$