STAT 516 Lec 06

Two-way factorial design (balanced, with replication)

Karl Gregory

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Tensile strength example from Kuehl (2000)

$n_{ij} = n = 3$ for all i, j.

Table 6.3 Tensile strength (psi) of asphaltic concrete specimens for two aggregate types with each of four compaction methods

Factor B: 4 levels ($b=4$)

Factor A
2 levels ($a=2$)

Aggregate Type	*Compaction Method*			
		Kneading		
	Static	*Regular*	*Low*	*Very Low*
Basalt	68	126	93	56
	63	128	101	59
	65	133	98	57
Silicious	71	107	63	40
	66	110	60	41
	66	116	59	44

Basalt, Static

$2 * 4 = 8$ treatments

$8 = ab$.

Source: A. M. Al-Marshed (1981), Compaction effects on asphaltic concrete durability. M.S. thesis, Civil Engineering, University of Arizona.

```
y <- c(68,63,65,71,66,66,126,128,133,107,110,116,
       93,101,98,63,60,59,56,59,57,40,41,44)
agg <- as.factor(rep(c(rep("B",3),rep("S",3)),4))
comp <- as.factor(c(rep("st",6),rep("r",6),
                    rep("l",6),rep("vl",6)))
```

The two-way factorial experimental design

- Two factors of interest.
- Each factor comprehends a set of treatments, called its levels.
- EUs randomly assigned to treatments.
- Each treatment is a unique combinations of factor levels.
- If Factor A has a levels and Factor B has b levels.
- There are ab treatment groups.

We want to make inferences about:

1. The effects of each factor.
2. The interactions between the two factors.
3. Various differences in treatment group means.

Discuss: Give factors and their levels in the tensile strength experiment.

Main effects and interactions

A main effect is an effect of a factor which does not depend on the level of the other factor.

An interaction is any dependence on the effect of one factor on the level of the other factor.

One-way model: $Y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, $\varepsilon_{ij} \sim N(0, \sigma^2)$

treatment mean i

$i = 1, \ldots, a$, $j = 1, \ldots, n$

Two-way treatment effects model

i = level of Factor A
j = level of Factor B
k = EU within treatment group.

Suppose the responses arise as

"gamma"

$$\underline{Y_{ijk}} = \mu + \tau_i + \underline{\gamma_j} + (\tau\gamma)_{ij} + \underline{\varepsilon_{ijk}}$$

for $i = 1, \ldots, a$, $j = 1, \ldots, b$, and $k = 1, \ldots, n_{ij}$, where

- Y_{ijk} is the response for EU k under level i of A and level j of B.
- μ represents a baseline or overall mean
- The τ_i are the main effects for Factor A.
- The γ_j are the main effects for Factor B.
- The $(\tau\gamma)_{ij}$ are the interaction effects between A and B.
- The ε_{ijk} are Normal$(0, \sigma^2)$ error terms.

Assume for now

1. a balanced design, i.e. $n_{ij} = n$ for all i, j
2. with replication, i.e. $n \geq 2$.

Parameter constraints in the treatment effects model

$$\underbrace{\mu}_{1},\ \underbrace{\tau_1, \dots, \tau_a}_{a},\ \underbrace{\gamma_1, \dots, \gamma_b}_{b},\ \underbrace{(\tau\gamma)_{11}, \dots, (\tau\gamma)_{ab}}_{ab}$$

Treatment effects model has $\boxed{a + b + ab + 1}$ parameters for $\boxed{ab}$ means...

To identify the parameters uniquely, we impose one of these constraints:

1. To give μ a baseline interpretation, set

$$\tau_a = 0, \quad \gamma_b = 0, \quad \text{and} \quad (\tau\gamma)_{aj} = (\tau\gamma)_{ib} = 0 \text{ for all } i, j.$$

2. To give μ an overall mean interpretation, set

$$\sum_{i=1}^{a} \tau_i = 0, \quad \sum_{j=1}^{b} \gamma_j = 0, \quad \sum_{j=1}^{b} (\tau\gamma)_{ij} = 0 \quad \forall i, \quad \sum_{i=1}^{a} (\tau\gamma)_{ij} = 0 \quad \forall j.$$

$$Y_{ijk} = \underbrace{\mu + \tau_i + \gamma_j + (\tau\gamma)_{ij}}_{\text{means}} + \varepsilon_{ijk}$$

Exercise: Let $a = 2$ and $b = 3$. Can tabulate the response means as:

	1	2	3
	$i=1, j=1$	$i=1, j=2$	
1	$\mu+\tau_1+\gamma_1+(\tau\gamma)_{11}$	$\mu+\tau_1+\gamma_2+(\tau\gamma)_{12}$	$\mu+\tau_1+\gamma_3+(\tau\gamma)_{13}$
2	$\mu+\tau_2+\gamma_1+(\tau\gamma)_{21}$	$\mu+\tau_2+\gamma_2+(\tau\gamma)_{22}$	$\mu+\tau_2+\gamma_3+(\tau\gamma)_{23}$

Rewrite the table under the *μ-as-baseline* constraints

$\tau_2 = 0$ $\quad$ $\gamma_3 = 0$

$\tau_a = 0$, $\gamma_b = 0$, and $\boxed{(\tau\gamma)_{aj} = (\tau\gamma)_{ib}} = 0$ for all i, j.

	1	2	3
	$i=1, j=1$	$i=1, j=2$	
1	$\mu+\tau_1+\gamma_1+(\tau\gamma)_{11}$	$\mu+\tau_1+\gamma_2+(\tau\gamma)_{12}$	$\mu+\tau_1+$ [redacted] $+($ [redacted]
2	$\mu+$ [redacted] $+\gamma_1+($ [redacted]	$\mu+$ [redacted] $+\gamma_2+($ [redacted]	$\mu+$ [redacted] $+$ [redacted] $+($ [redacted]

Only parameters left:

$6 = 2 * 3$

μ = mean for the (2, 3) treatment group

τ_1

γ_1

γ_2

$(\tau\gamma)_{11}$

$(\tau\gamma)_{12}$

Cell means model representation

Assume

$\mu + \tau_i + \gamma_j + (\tau\gamma)_{ij}$

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$

mean for treatment i,j.

for $i = 1, \ldots, a$, $j = 1, \ldots, b$, and $k = 1, \ldots, n_{ij}$, where

- Y_{ijk} is the response for EU k under level i of A and level j of B.
- μ_{ij} is the response mean under level i of A and level j of B.
- The ε_{ijk} are Normal$(0, \sigma^2)$ error terms.

Define the marginal means

mean at level j of factor B

$$\bar{\mu}_{i.} = \frac{1}{b}\sum_{j=1}^{b} \mu_{ij}, \quad i = 1, \ldots, a \quad \text{and} \quad \bar{\mu}_{.j} = \frac{1}{a}\sum_{i=1}^{a} \mu_{ij}, \quad j = 1, \ldots, b.$$

mean at level i of factor A

and the overall mean $\bar{\mu}_{..} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \mu_{ij}$.

Hypotheses of interest in the two-way factorial experiment

State hypothesis for the cell means model $Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$.

$$SS_A = nb\sum_{i=1}^{a}(\bar{Y}_{i..} - \bar{Y}_{..})^2$$

1. H_0: Factor A has no main effect.

$$H_0\colon\ \bar{\mu}_{1.} = \cdots = \bar{\mu}_{a.}.$$

large SSA carries evidence against this

2. H_0: Factor B has no main effect.

$$H_0\colon\ \bar{\mu}_{.1} = \cdots = \bar{\mu}_{.b}.$$

large SS_B carries evidence against this

3. H_0: There is no interaction between Factor A and Factor B.

$$H_0\colon\ \mu_{ij} = \bar{\mu}_{i.} + \bar{\mu}_{.j} - \bar{\mu}_{..} \quad \text{for all } i, j.$$

Large $SSAB$ carries evidence against this.

No interaction means $\mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$

$a = 2,\ b = 3$

	1	2	3
1	μ_{11}	μ_{12}	μ_{13}
2	μ_{21}	μ_{22}	μ_{23}

=

$\bar{\mu}_{1\cdot} + \bar{\mu}_{\cdot 1} - \bar{\mu}_{\cdot\cdot}$	$\bar{\mu}_{1\cdot} + \bar{\mu}_{\cdot 2} - \bar{\mu}_{\cdot\cdot}$	$\bar{\mu}_{1\cdot} + \bar{\mu}_{\cdot 3} - \bar{\mu}_{\cdot\cdot}$	$\bar{\mu}_{1\cdot}$
$\bar{\mu}_{2\cdot} + \bar{\mu}_{\cdot 1} - \bar{\mu}_{\cdot\cdot}$	$\bar{\mu}_{2\cdot} + \bar{\mu}_{\cdot 2} - \bar{\mu}_{\cdot\cdot}$	$\bar{\mu}_{2\cdot} + \bar{\mu}_{\cdot 3} - \bar{\mu}_{\cdot\cdot}$	$\bar{\mu}_{2\cdot}$
$\bar{\mu}_{\cdot 1}$	$\bar{\mu}_{\cdot 2}$	$\bar{\mu}_{\cdot 3}$	

$$\left.\begin{aligned} \mu_{11} - \mu_{21} &= \bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} \\ \mu_{12} - \mu_{22} &= \bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} \\ \mu_{13} - \mu_{23} &= \bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} \end{aligned}\right)$$

same effect of Factor A across all levels of Factor B.

Avg of first row $= \dfrac{(\bar{\mu}_{1\cdot} + \bar{\mu}_{\cdot 1} - \bar{\mu}_{\cdot\cdot}) + (\bar{\mu}_{1\cdot} + \bar{\mu}_{\cdot 2} - \bar{\mu}_{\cdot\cdot}) + (\bar{\mu}_{1\cdot} + \bar{\mu}_{\cdot 3} - \bar{\mu}_{\cdot\cdot})}{3}$

$$= \bar{\mu}_{1\cdot} + \underbrace{\frac{1}{3}\left(\bar{\mu}_{\cdot 1} + \bar{\mu}_{\cdot 2} + \bar{\mu}_{\cdot 3}\right)}_{\bar{\mu}_{\cdot\cdot}} - \bar{\mu}_{\cdot\cdot}$$

$$= \bar{\mu}_{1\cdot}$$

Factor A no main effect:

$$H_0: \bar{\mu}_{1\cdot} = \bar{\mu}_{2\cdot}$$

Factor B no main effect:

$$H_0: \bar{\mu}_{\cdot 1} = \bar{\mu}_{\cdot 2} = \bar{\mu}_{\cdot 3}$$

Example: Let $a = 2$ and $b = 3$. Can tabulate the response means as:

Factor B

Factor A	1	2	3	
1	μ_{11}	μ_{12}	μ_{13}	$\bar{\mu}_{1\cdot} = (\mu_{11} + \mu_{12} + \mu_{13})/3$
2	μ_{21}	μ_{22}	μ_{23}	$\bar{\mu}_{2\cdot} = (\mu_{21} + \mu_{22} + \mu_{23})/3$
	$\bar{\mu}_{\cdot 1}$	$\bar{\mu}_{\cdot 2}$	$\bar{\mu}_{\cdot 3}$	$\bar{\mu}_{\cdot\cdot} = \frac{(\mu_{11} + \mu_{12} + \mu_{13} + \mu_{21} + \mu_{22} + \mu_{23})}{2 \cdot 3}$

Make sense of the hypotheses of no main effects and of no interaction.

$$\frac{(\mu_{11} + \mu_{21})}{2} \qquad \frac{(\mu_{12} + \mu_{22})}{2} \qquad \frac{(\mu_{13} + \mu_{23})}{2}$$

Goals in two-way factorial experiments

In the two-way treatment effects model

Factor A

Interaction effect

$$Y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + \varepsilon_{ijk}, \quad = \mu_{ij} + \varepsilon_{ijk}$$

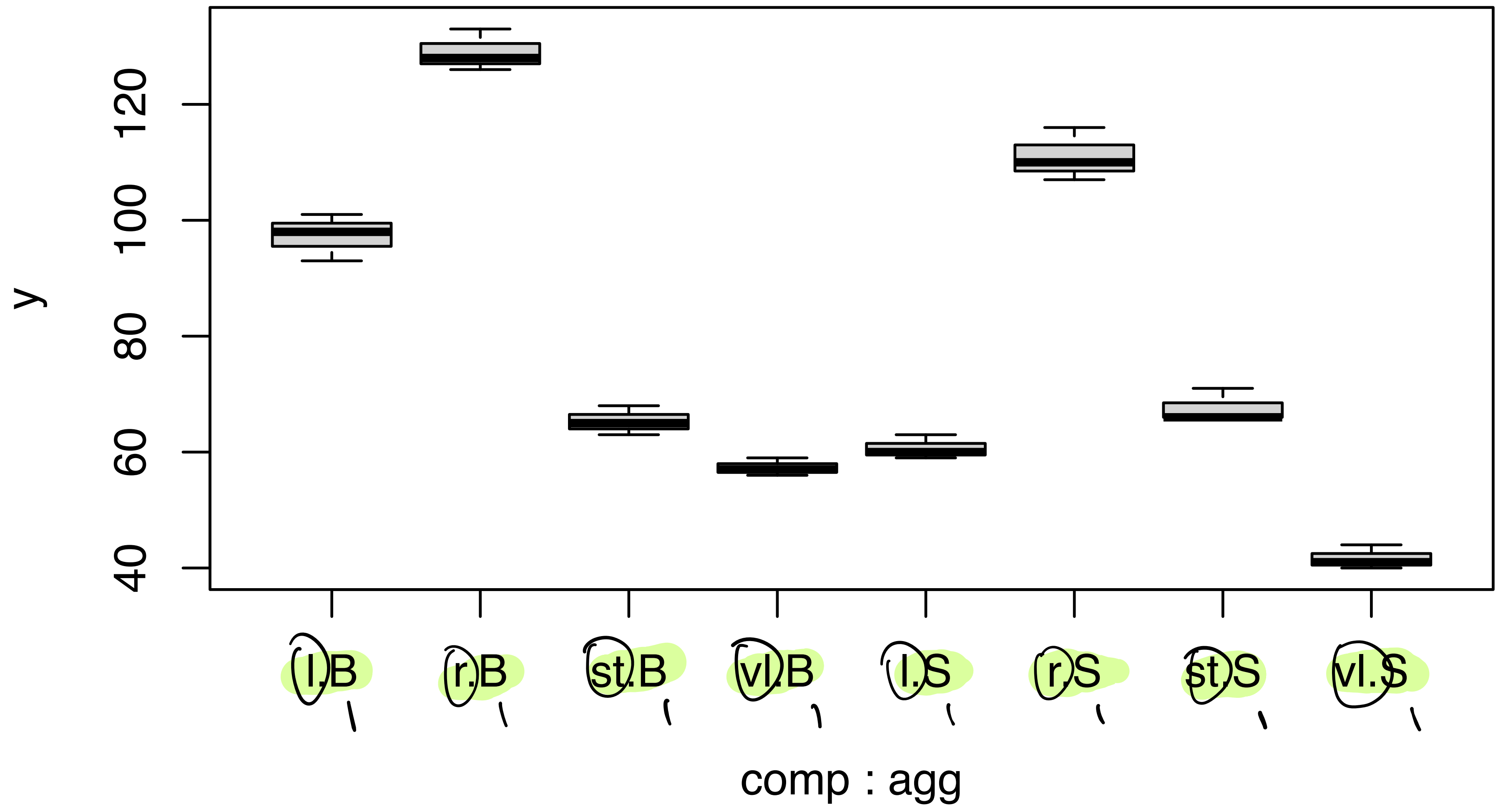
Factor B

where $\varepsilon_{ijk} \overset{\text{ind}}{\sim}$ Normal$(0, \sigma^2)$, we wish to

1. Visualize the data.
2. Estimate the parameters μ, τ_i, γ_j, $(\tau\gamma)_{ij}$.
3. Estimate the error term variance σ^2.
4. Check whether the model assumptions are satisfied.
5. Decompose the variation in the Y_{ijk} into its sources.
6. Test whether there is *any* effect of the factors on the response.
7. Test for main effects and interaction effects.
8. Do multiple comparisons.

Tensile strength data (cont)

```
boxplot(y ~ comp:agg)
```



Tensile example (cont)

Use `summary()` on the `lm()` function output.

```
lm_out <- lm(y ~ agg + comp + agg:comp)
summary(lm_out)
```

$$Y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + \varepsilon_{ijk}$$

$\tau_B = 0$

$(\tau\gamma)_{B,r} = (\tau\gamma)_{B,\ell} = (\tau\gamma)_{B,vl} = (\tau\gamma)_{B,st} = 0$

$(\tau\gamma)_{S,\ell} = 0$

μ-as-baseline constraint.

```
Call:
lm(formula = y ~ agg + comp + agg:comp)

Residuals:
    Min      1Q  Median      3Q     Max
-4.3333 -1.6667 -0.6667  2.3333  5.0000

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    97.333      1.780  54.697  < 2e-16 ***
aggS          -36.667      2.517 -14.570 1.18e-10 ***
compr          31.667      2.517  12.583 1.03e-09 ***
compst        -32.000      2.517 -12.716 8.85e-10 ***
compvl        -40.000      2.517 -15.894 3.20e-11 ***
aggS:compr     18.667      3.559   5.245 8.01e-05 ***
aggS:compst    39.000      3.559  10.958 7.58e-09 ***
aggS:compvl    21.000      3.559   5.900 2.24e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.082 on 16 degrees of freedom
Multiple R-squared:  0.9921,    Adjusted R-squared:  0.9887
F-statistic: 287.6 on 7 and 16 DF,  p-value: 1.305e-15
```

$\hat{\mu}$, $\hat{\tau}_S$, $\hat{\gamma}_r$, $\hat{\gamma}_{st}$, $\hat{\gamma}_{vl}$, $\widehat{(\tau\gamma)}_{S,r}$, $\widehat{(\tau\gamma)}_{S,st}$, $\widehat{(\tau\gamma)}_{S,vl}$

parameter estimates

Exercise: See how the parameter estimates build the treatment means.

```
aggregate(y, by = list(agg, comp), FUN = mean)
```

```
  Group.1 Group.2         x
1       B       l  97.33333
2       S       l  60.66667
3       B       r 129.00000
4       S       r 111.00000
5       B      st  65.33333
6       S      st  67.66667
7       B      vl  57.33333
8       S      vl  41.66667
```

$\hat{\mu} + \hat{\tau}_S = 97.33 + (-36.667)$

$\hat{\mu} + \hat{\gamma}_r = 97.33 + 31.667$

treatment group means.

$$\bar{Y}_{ij\cdot} = \hat{\mu} + \hat{\tau}_i + \hat{\gamma}_j + \widehat{(\tau\gamma)}_{ij}$$

The fitted values

Y_{ijk}

$a=2$

$b=3$

$\bar{Y}_{11.}$	$\bar{Y}_{12.}$	$\bar{Y}_{13.}$
$\bar{Y}_{21.}$	$\bar{Y}_{22.}$	$\bar{Y}_{23.}$

Define the treatment group means

$$\bar{Y}_{ij.} = \frac{1}{n}\sum_{k=1}^{n} Y_{ijk} \quad \text{for } i = 1, \dots, a, \quad j = 1, \dots, b.$$

Then the

- fitted values are the treatment group means, i.e. $\hat{Y}_{ijk} = \bar{Y}_{ij.}$.
- residuals are the deviations from the group means $\hat{\varepsilon}_{ijk} = Y_{ijk} - \bar{Y}_{ij.}$.

Estimating the error term variance

$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}, \quad \varepsilon_{ijk} \sim N(0, \sigma^2)$

An unbiased estimator of the error term variance σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{ab(n-1)} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} \overbrace{(Y_{ijk} - \bar{Y}_{ij.})^2}^{\hat{\varepsilon}_{ijk}^2},$$

which is the sum of the squared residuals divided by $ab(n-1)$.

$$ab(n-1) = \underbrace{abn}_{\text{Total \# of EUs / observations}} - \underbrace{ab}_{\text{\# treatment groups}}$$

Checking model assumptions

$$y_{ijk} = \underbrace{\mu + \tau_i + \beta_j + (\tau\beta)_{ij}}_{\mu_{ij}} + \varepsilon_{ijk}, \qquad \varepsilon_{ijk} \sim N(0, \sigma^2)$$

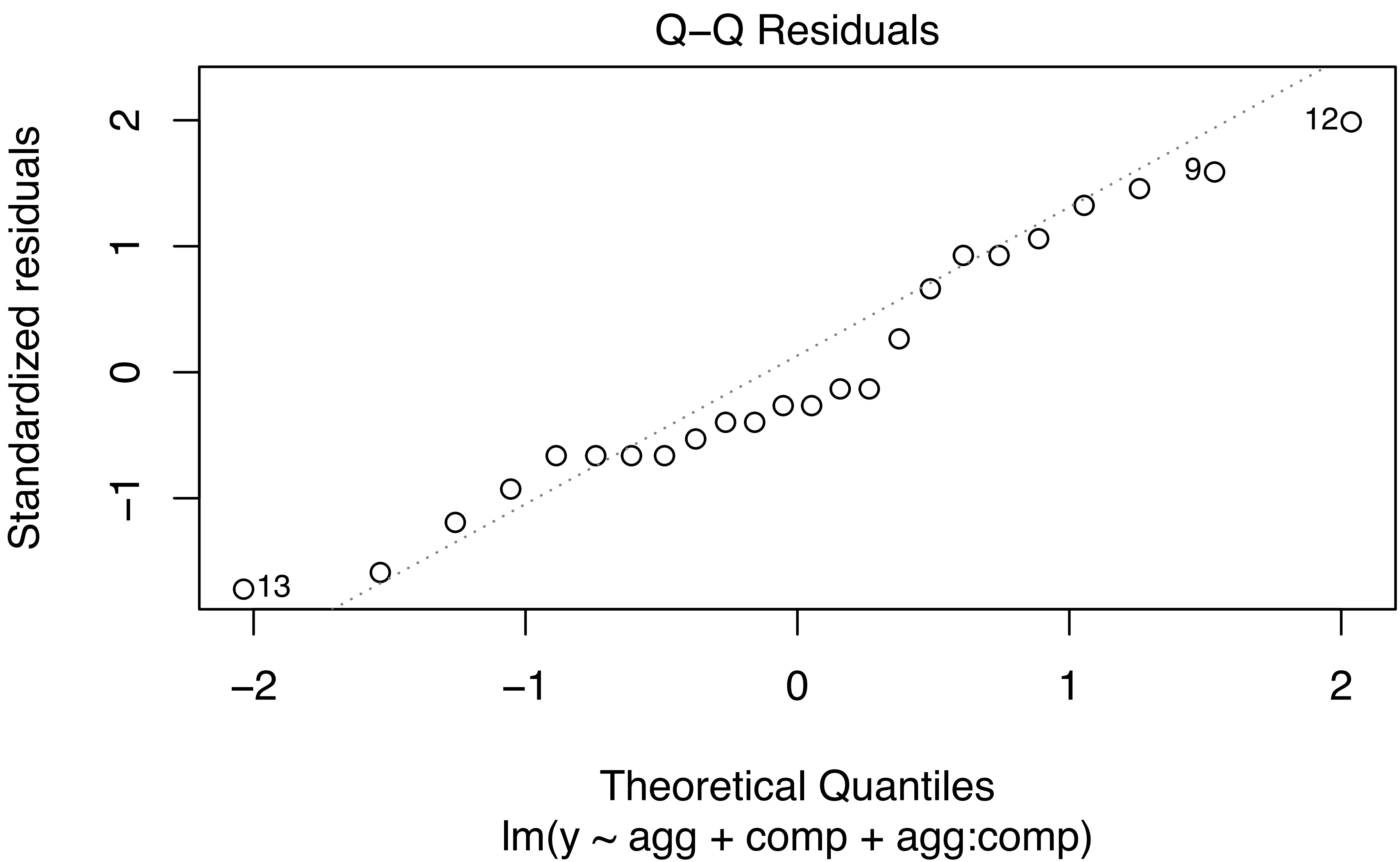
$$y_{ijk} = \mu_{ij} + \varepsilon_{ijk},$$

Validity of the methods in these slides depends on these assumptions:

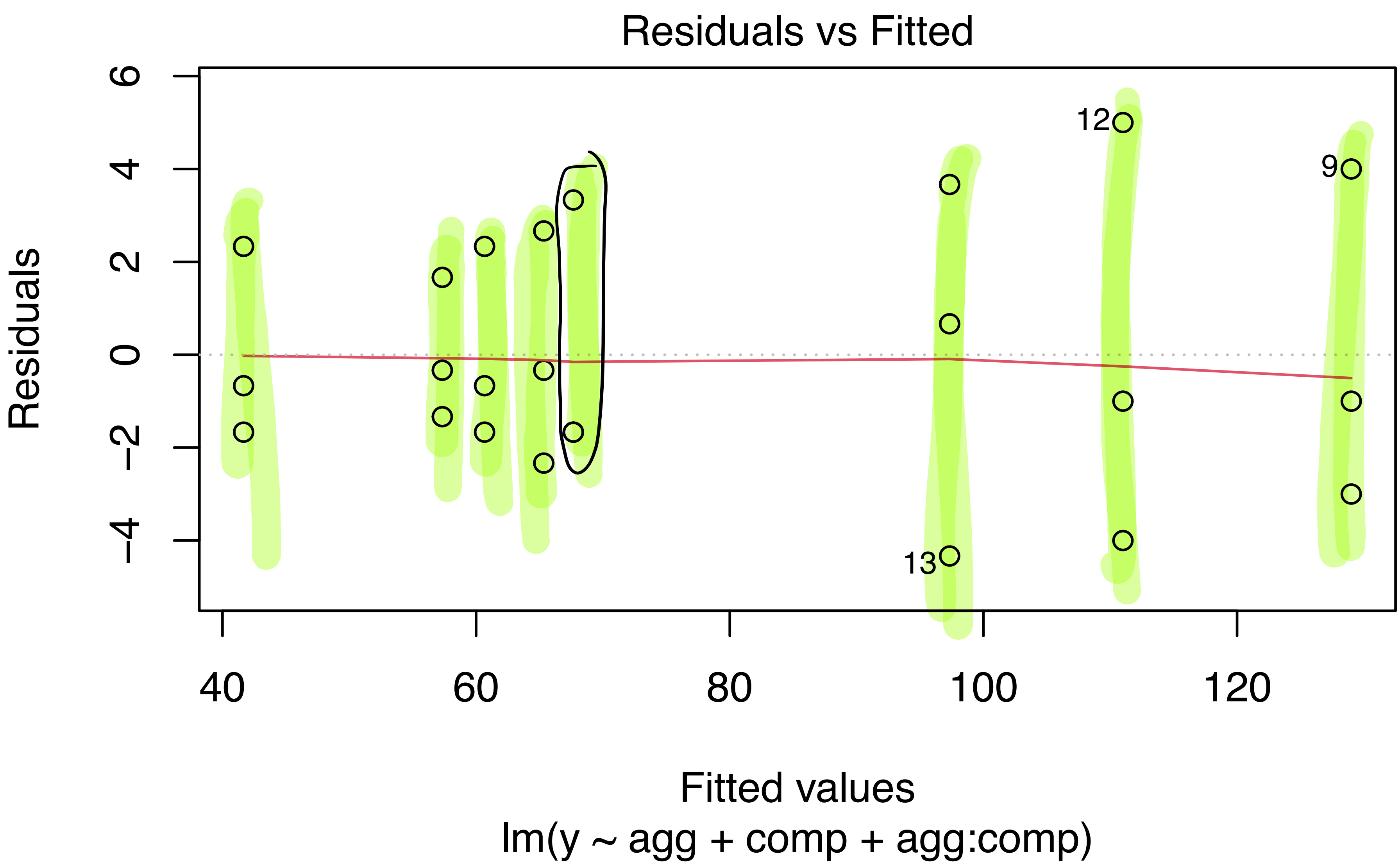
1. The responses are normally distributed around the treatment means (Check QQ plot of residuals).
2. The response has the same variance in all treatment groups (Check residuals vs fitted values plot).
3. The response values are independent of each other (No way to check; must trust experimental design).

Tensile example (cont)

```
plot(lm_out,which = 2)
```



```r
plot(lm_out,which = 1)
```



Sums of squares in the two-way factorial experiment

$\bar{Y}_{\dots} = \frac{1}{abn} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} Y_{ijk}$

As in linear regression we decompose the variation in the Y_{ijk} by defining:

- Total sum of squares: $\mathrm{SS_{Tot}} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (Y_{ijk} - \bar{Y}_{\dots})^2$
- Treatments sum of squares: $\mathrm{SS_{Trt}} = \sum_{i=1}^{a} \sum_{j=1}^{b} n(\bar{Y}_{ij.} - \bar{Y}_{\dots})^2$
- Error sum of squares: $\mathrm{SS_{Error}} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} \underbrace{(Y_{ijk} - \bar{Y}_{ij.})^2}_{\hat{\varepsilon}_{ijk}^2}$

We have $\boxed{\mathrm{SS_{Tot}}} = \boxed{\mathrm{SS_{Trt}} + \mathrm{SS_{Error}}}$.

We again define $R^2 = \dfrac{\mathrm{SS_{Trt}}}{\mathrm{SS_{Tot}}}$.

$$SS_{Trt} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\hat{Y}_{ijk} - \bar{Y}_{\dots})^2 = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\bar{Y}_{ij.} - \bar{Y}_{..})^2 = \sum_{i=1}^{a} \sum_{j=1}^{b} n (\bar{Y}_{ij.} - \bar{Y}_{\dots})^2$$

$\bar{Y}_{ij.} = \frac{1}{n} \sum_{k=1}^{n} Y_{ijk}$

Sampling distributions of our sums of squares

The SS, appropriately scaled, follow chi-square distributions:

- $\mathrm{SS}_{\mathrm{Tot}}/\sigma^2 \sim \chi^2_{abn-1}(\phi_{\mathrm{Tot}})$
- $\mathrm{SS}_{\mathrm{Trt}}/\sigma^2 \sim \chi^2_{ab-1}(\phi_{\mathrm{Trt}})$
- $\mathrm{SS}_{\mathrm{Error}}/\sigma^2 \sim \chi^2_{ab(n-1)}$,

where ϕ_{Tot} and ϕ_{Trt} are noncentrality parameters.

The mean squares in the one-way ANOVA model

Dividing $\mathrm{SS}_{\mathrm{Trt}}$ and $\mathrm{SS}_{\mathrm{Error}}$ by their dfs, we define:

- Treatments mean square: $\mathrm{MS}_{\mathrm{Trt}} = \dfrac{\mathrm{SS}_{\mathrm{Trt}}}{ab-1}$
- Error mean square: $\mathrm{MS}_{\mathrm{Error}} = \dfrac{\mathrm{SS}_{\mathrm{Error}}}{ab(n-1)} = \hat{\sigma}^2$

The ratio $F_{\mathrm{stat}} = \dfrac{\mathrm{MS}_{\mathrm{Trt}}}{\mathrm{MS}_{\mathrm{Error}}}$ has an F distribution.

The Analysis of Variance (ANOVA) table

$SS_{TRT} = \sum_{i=1}^{a} \sum_{j=1}^{b} n (\bar{Y}_{ij\cdot} - \bar{Y}_{\cdots})^2$

sum of ab things, must first compute 1 overall mean

We often present the SS, df, and MS values in a table like this:

$SS_A + SS_B + SS_{AB}$

Source	Df	SS	MS	F value	p-value
Treatments	$ab - 1$	$\mathrm{SS}_{\mathrm{Trt}}$	$\mathrm{MS}_{\mathrm{Trt}}$	F_{stat}	$P(F > F_{\mathrm{stat}})$
Error	$ab(n-1)$	$\mathrm{SS}_{\mathrm{Error}}$	$\mathrm{MS}_{\mathrm{Error}}$		
Total	$\underbrace{abn}_{\text{total \# observations}} - 1$	$\mathrm{SS}_{\mathrm{Tot}}$			

In the table $F_{\mathrm{stat}} = \dfrac{\mathrm{MS}_{\mathrm{Trt}}}{\mathrm{MS}_{\mathrm{Error}}}$.

The p-value is based on $F \sim F_{ab-1,ab(n-1)}$.

Exercise: Fill in the ANOVA table for the tensile strength data.

```
a <- 2
b <- 4
n <- 3
yhat <- predict(lm_out)
ehat <- y - yhat
SSE <- sum(ehat^2)
MSE <- SSE / (a*b*(n-1))
SSR <- sum((yhat - mean(y))^2)
MSR <- SSR / (a*b - 1)
SST <- sum((y - mean(y))^2)
Fstat <- MSR / MSE
pval <- 1 - pf(Fstat,a*b-1,a*b*(n-1))
```

$a = 2$
$b = 4$) Tensile strength data
$n = 3$

$2 \cdot 4 - 1 = 8 - 1 = 7$

Source	Df	SS	MS	F value	p-value
Treatments	7	19122.50	2731.79	287.5564	0.0000
Error	16	152.00	9.50		
Total	23	19274.50			

$2 \cdot 4 \cdot 3 - 1 = 24 - 1$

Reject $H_0: \mu_{ij}$ all the same

$2 \cdot 4(3-1) = 8 \cdot 2 = 16$

Overall F test for any kind of effect

Treatment effects representation   Cell means representation.

$$Y_{ijk} = \underbrace{\mu + \tau_i + \gamma_j + (\tau\gamma)_{ij}}_{} + \varepsilon_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$

Using `summary()` on the `lm()` output prints the overall F test result.

The overall F test in the two-way treatment effects model tests

H_0: All the μ_{ij} are the same.

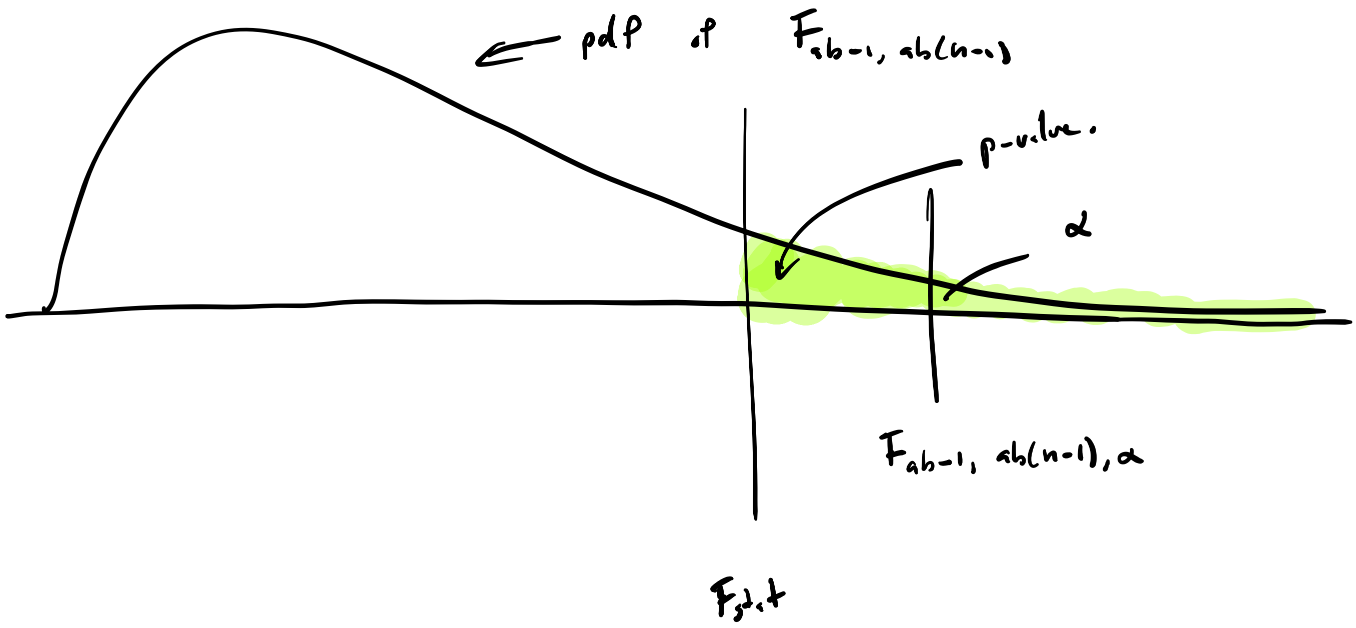
H_1: The μ_{ij} are not all the same.

The null says neither factor has any effect on the mean response.

The alternate says at least one of the factors has some effect.

Exercise: Interpret overall F test result for the tensile strength data.

Use $F_{stat} = \dfrac{MS_{Trt}}{MS_{Error}} \overset{H_0}{\sim} F_{ab-1,\, ab(n-1)}$



Reject H_0: μ_{ij} all the same if $F_{stat} > F_{ab-1, ab(n-1), \alpha}$.

Treatment effects representation — Cell means representation.

$$Y_{ijk} = \underbrace{\mu + \tau_i + \beta_j + (\tau\beta)_{ij}}_{\text{comp}} + \varepsilon_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$

	st	r	l	vl
B	μ_{11}	μ_{12}	μ_{13}	μ_{14}
S	μ_{21}	μ_{22}	μ_{23}	μ_{24}

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}, \quad \varepsilon_{ijk} \sim N(0, \sigma^2)$$

$$\mu_{ij} \to \mu + \tau_i + \beta_j + (\tau\beta)_{ij}$$

$$SS_{Trt} = \sum_{i=1}^{a} \sum_{j=1}^{b} n \left(\bar{Y}_{ij.} - \bar{Y}_{...} \right)^2$$

How much do the trt means vary around the overall mean?

Further decomposition of treatments sum of squares

Main effects   Interaction effect

Consider the decomposition $\mathrm{SS_{Trt}} = \underbrace{\mathrm{SS_A} + \mathrm{SS_B}}_{\text{Main effects}} + \underbrace{\mathrm{SS_{AB}}}_{\text{Interaction effect}}$, where

sum of a squares, depending on $\bar{Y}_{\dots}$

- $\mathrm{SS_A} = bn\sum_{i=1}^{a}\boxed{(\bar{Y}_{i..} - \bar{Y}_{...})^2}$ (Factor A SS)   Variation in Factor A means around overall mean.
- $\mathrm{SS_B} = an\sum_{j=1}^{b}(\bar{Y}_{.j.} - \bar{Y}_{...})^2$ (Factor B SS)   Variation in Factor B means around overall mean.
- $\mathrm{SS_{AB}} = n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{ij.} - (\bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}))^2$ (Interaction SS)

Measures evidence in factor of interaction.

For some noncentrality parameters ϕ_A, ϕ_B, and ϕ_{AB}, we have

- $\mathrm{SS_A}/\sigma^2 \sim \chi^2_{a-1}(\phi_A)$
- $\mathrm{SS_B}/\sigma^2 \sim \chi^2_{b-1}(\phi_B)$
- $\mathrm{SS_{AB}}/\sigma^2 \sim \chi^2_{(a-1)(b-1)}(\phi_{AB})$.

SS_{AB} is sum of ab terms, need to compute $\bar{Y}_{i..}$, $\bar{Y}_{.j.}$ and $\bar{Y}_{..}$

$MS_{TRT} \neq MS_A + MS_B + MS_{AB}$

Define the corresponding mean squares

$$\mathrm{MS_A} = \frac{\mathrm{SS_A}}{a-1}, \quad \mathrm{MS_B} = \frac{\mathrm{SS_B}}{b-1}, \quad \mathrm{MS_{AB}} = \frac{\mathrm{SS_{AB}}}{(a-1)(b-1)}.$$

$SS_{AB} = n\sum_{i=1}^{a}\sum_{j=1}^{b}\Big(\overset{ab-1}{\bar{Y}_{ij.} - \bar{Y}_{..}} - \overbrace{(\bar{Y}_{i..} - \bar{Y}_{...})}^{a-1} - \overbrace{(\bar{Y}_{.j.} - \bar{Y}_{..})}^{b-1}\Big)^2$, has df $(ab-1) - (a-1) - (b-1)$
$= ab - a - b + 1$

$= (a-1)(b-1)$

Full ANOVA table for balanced two-way factorial design

numerator degrees of freedom.

We've decomposed SS_{Trt} into $SS_A + SS_B + SS_{AB}$

Source	Df	SS	MS	F value
A	$a-1$	$\mathrm{SS_A}$	$\mathrm{MS_A}$	$F_\mathrm{A} = \mathrm{MS_A} / \mathrm{MS_{Error}}$
B	$b-1$	$\mathrm{SS_B}$	$\mathrm{MS_B}$	$F_\mathrm{B} = \mathrm{MS_B} / \mathrm{MS_{Error}}$
AB	$(a-1)(b-1)$	$\mathrm{SS_{AB}}$	$\mathrm{MS_{AB}}$	$F_\mathrm{AB} = \mathrm{MS_{AB}} / \mathrm{MS_{Error}}$
Error	$ab(n-1)$	$\mathrm{SS_{Error}}$	$\mathrm{MS_{Error}}$	
Total	$abn-1$	$\mathrm{SS_{Tot}}$		

denominator of F_{stat}

den df

$\bar{\mu}_{1\cdot} = \cdots = \bar{\mu}_{a\cdot}$

1. Reject H_0: no Factor A main effect if $F_\mathrm{A} > F_{a-1,ab(n-1),\alpha}$.

$\bar{\mu}_{\cdot 1} = \cdots = \bar{\mu}_{\cdot b}$

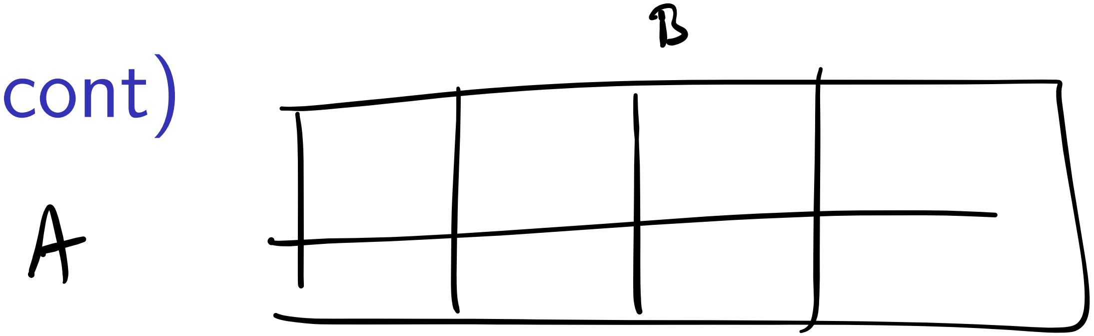
2. Reject H_0: no Factor B main effect if $F_\mathrm{B} > F_{b-1,ab(n-1),\alpha}$.
3. Reject H_0: no A and B interaction if $F_\mathrm{AB} > F_{(a-1)(b-1),ab(n-1),\alpha}$.

$\mu_{ij} = \bar{\mu}_{i\cdot} + \bar{\mu}_{\cdot j} - \bar{\mu}_{\cdot\cdot} \quad \forall\, ij$

Tensile strength data (cont)

$a=2 \quad n=3$

$b=4$



Obtain ANOVA table with `anova()` function on `lm()` output.

```
anova(lm(y ~ agg + comp + agg:comp))
```

$182.526 = \frac{1734.0}{9.5}$

```
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)    
agg        1   1734  1734.0 182.526 3.628e-10 ***
comp       3  16244  5414.5 569.947 < 2.2e-16 ***
agg:comp   3   1145   381.7  40.175 1.124e-07 ***
Residuals 16    152     9.5                      
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

A, B, AB, Error, Total

$a-1=2-1=1$

$b-1=4-1=3$

MS_{Error}

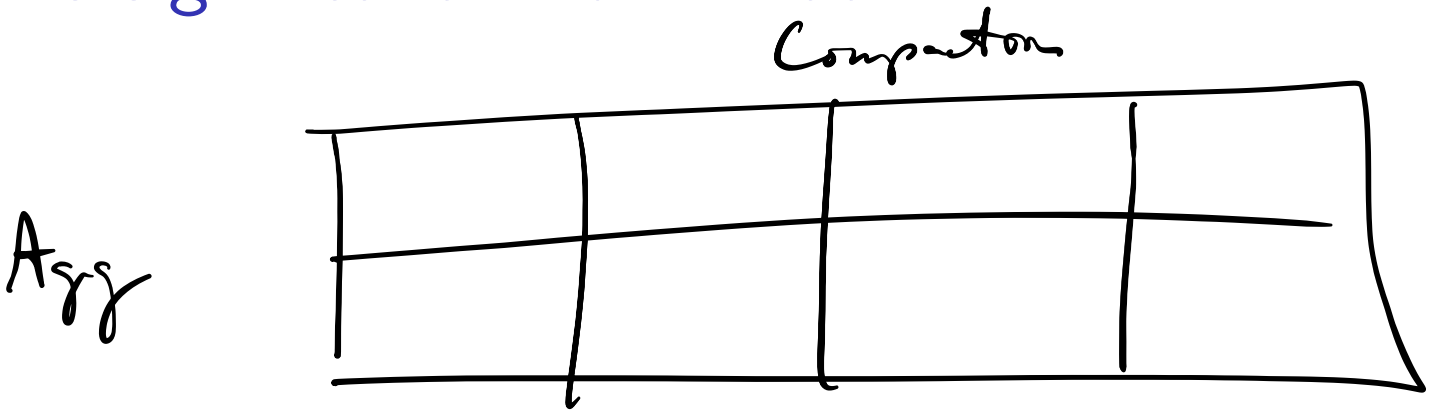
Reject H_0 for 1. 2. and 3. on previous slide.

$(a-1)(b-1)=(2-1)(4-1)=3$

Important: `anova()` function only appropriate for a balanced design.

$ab(n-1) = 2\cdot 4(3-1) = 16$

Interaction is significant. Now what?



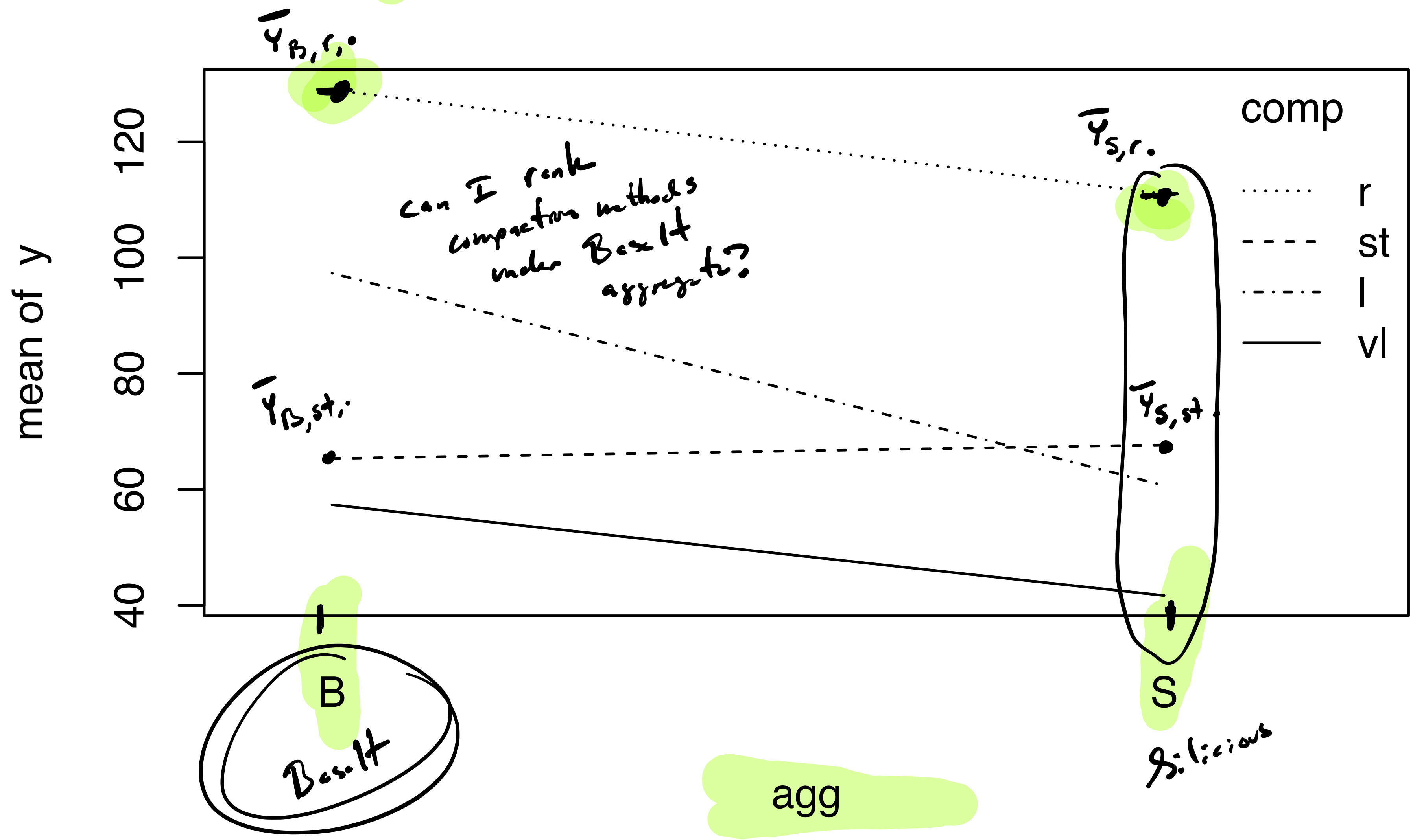
When you find a significant interaction:

1. Make interaction plots (next slides).
2. Be very cautious about interpreting main effects, even when these are statistically significant.

Tensile strength data (cont)

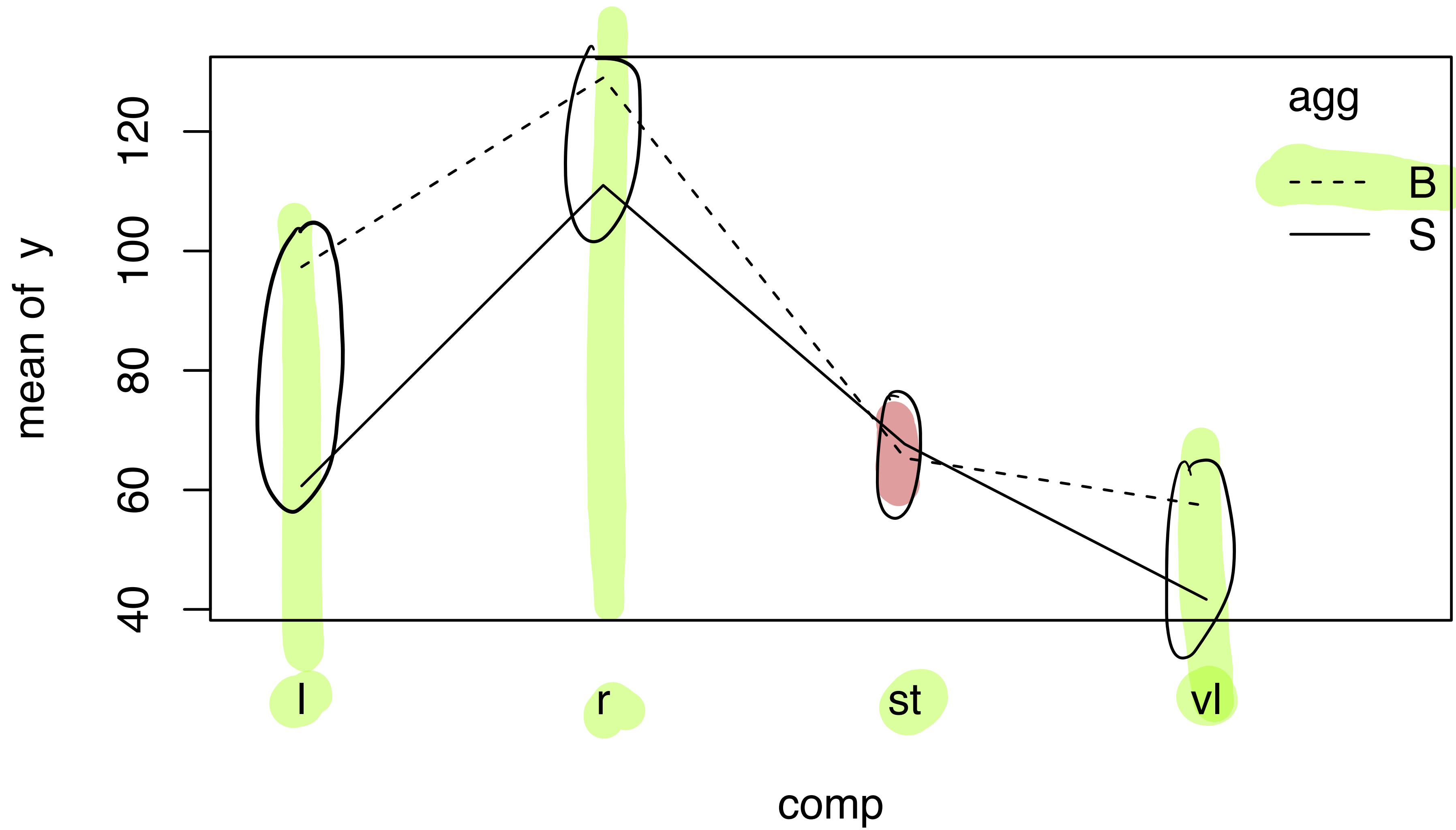
Use the `interaction.plot()` function to visualize an interaction:

```
interaction.plot(agg,comp,y)
```



Interactions appear as crossing lines or differing slopes.

```r
interaction.plot(comp, agg, y)
```



Estimates of cell and marginal means in the balanced case

$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$

μ_{11}	μ_{12}	μ_{13}	μ_{14}	$\bar{\mu}_{1\cdot}$
μ_{21}	μ_{22}	μ_{23}	μ_{24}	$\bar{\mu}_{2\cdot}$
$\bar{\mu}_{\cdot 1}$	$\bar{\mu}_{\cdot 2}$	$\bar{\mu}_{\cdot 3}$	$\bar{\mu}_{\cdot 4}$	$\bar{\mu}_{\cdot\cdot}$

marginal means $\longrightarrow$

The estimators of the cell and marginal means are given by

- $\hat{\mu}_{ij} = \bar{Y}_{ij.}$, $i = 1, \ldots, a$, $j = 1, \ldots, b$
- $\hat{\bar{\mu}}_{i.} = \bar{Y}_{i..}$, $i = 1, \ldots, a$.
- $\hat{\bar{\mu}}_{.j} = \bar{Y}_{.j.}$, $j = 1, \ldots, b$.

We estimate $\hat{\bar{\mu}}_{i.}$ with $\bar{Y}_{i..}$ (and $\hat{\bar{\mu}}_{.j}$ with $\bar{Y}_{.j.}$) only when $n_{ij} = n\ \forall ij$.

$\hat{\mu}_{ij} = \bar{Y}_{ij\cdot}$

$\hat{\bar{\mu}}_{i\cdot} = \bar{Y}_{i\cdot\cdot}$

$\hat{\bar{\mu}}_{\cdot j} = \bar{Y}_{\cdot j\cdot}$

$\bar{Y}_{11\cdot}$	$\bar{Y}_{12\cdot}$	$\bar{Y}_{13\cdot}$	$\bar{Y}_{14\cdot}$	$\bar{Y}_{1\cdot\cdot}$
$\bar{Y}_{21\cdot}$	$\bar{Y}_{22\cdot}$	$\bar{Y}_{23\cdot}$	$\bar{Y}_{24\cdot}$	$\bar{Y}_{2\cdot\cdot}$
$\bar{Y}_{\cdot 1\cdot}$	$\bar{Y}_{\cdot 2\cdot}$	$\bar{Y}_{\cdot 3\cdot}$	$\bar{Y}_{\cdot 4\cdot}$	

Some CI formulas (without familywise adjustment)

These CI formulas are for the balanced design $n_{ij} = n \ \forall ij$.

ab(n-1) is df for SSError

Target	$(1-\alpha)100\%$ confidence interval
μ_{ij}	$\bar{Y}_{ij.} \pm t_{ab(n-1),\alpha/2}\hat{\sigma}\sqrt{\frac{1}{n}}$
$\mu_{ij} - \mu_{i'j'}$	$\bar{Y}_{ij.} - \bar{Y}_{i'j'.} \pm t_{ab(n-1),\alpha/2}\hat{\sigma}\sqrt{\frac{2}{n}}$
$\bar{\mu}_{i.}$ ← marginal mean for Factor A.	$\bar{Y}_{i..} \pm t_{ab(n-1),\alpha/2}\hat{\sigma}\sqrt{\frac{1}{bn}}$
$\bar{\mu}_{.j}$	$\bar{Y}_{.j.} \pm t_{ab(n-1),\alpha/2}\hat{\sigma}\sqrt{\frac{1}{an}}$
$\bar{\mu}_{i.} - \bar{\mu}_{i'.}$	$\bar{Y}_{i..} - \bar{Y}_{i'..} \pm t_{ab(n-1),\alpha/2}\hat{\sigma}\sqrt{\frac{2}{bn}}$
$\bar{\mu}_{.j} - \bar{\mu}_{.j'}$	$\bar{Y}_{.j.} - \bar{Y}_{.j'.} \pm t_{ab(n-1),\alpha/2}\hat{\sigma}\sqrt{\frac{2}{an}}$

In the above $\hat{\sigma} = \sqrt{\text{MS}_{\text{Error}}}$.

Comparing means at all factor level combinations

$\binom{8}{2}$ pairs of means.

μ_{11}	μ_{12}	μ_{13}	μ_{14}
μ_{21}	μ_{21}	μ_{23}	μ_{24}

- Tukey's for comparing all pairs among μ_{ij}, $i = 1, \ldots, a$, $j = 1, \ldots, b$:

$$\bar{Y}_{ij.} - \bar{Y}_{i'j'.} \pm \boxed{q_{ab,ab(n-1),\alpha}}\,\hat{\sigma}\frac{1}{\sqrt{n}}, \quad (i,j) \neq (i',j').$$

- Dunnett's for comparing all means μ_{ij} to a baseline μ_{ab}:

$$\bar{Y}_{ij.} - \bar{Y}_{ab.} \pm d_{ab,ab(n-1),\alpha}\,\hat{\sigma}\sqrt{\frac{2}{n}}, \quad (i,j) \neq (a,b).$$

Use $\hat{\sigma} = \sqrt{\mathrm{MS}_{\mathrm{Error}}}$.

Control

μ_{11}	μ_{12}	μ_{13}	μ_{14}
μ_{21}	μ_{21}	μ_{23}	μ_{24}

These may make more comparisons than are of interest...

8-1 comparisons

Tensile strength data (cont)

```
TukeyHSD(aov(lm(y ~ agg:comp)))
```

```
  Tukey multiple comparisons of means
    95% family-wise confidence level

Fit: aov(formula = lm(y ~ agg:comp))

$`agg:comp`
                diff        lwr         upr     p adj
S:l-B:l   -36.666667 -45.379555 -27.9537785 0.0000000
B:r-B:l    31.666667  22.953778  40.3795548 0.0000000
S:r-B:l    13.666667   4.953778  22.3795548 0.0011210
B:st-B:l  -32.000000 -40.712888 -23.2871118 0.0000000
S:st-B:l  -29.666667 -38.379555 -20.9537785 0.0000001
B:vl-B:l  -40.000000 -48.712888 -31.2871118 0.0000000
S:vl-B:l  -55.666667 -64.379555 -46.9537785 0.0000000
B:r-S:l    68.333333  59.620445  77.0462215 0.0000000
S:r-S:l    50.333333  41.620445  59.0462215 0.0000000
B:st-S:l    4.666667  -4.046222  13.3795548 0.5964603
S:st-S:l    7.000000  -1.712888  15.7128882 0.1678762
B:vl-S:l   -3.333333 -12.046222   5.3795548 0.8765993
S:vl-S:l  -19.000000 -27.712888 -10.2871118 0.0000257
S:r-B:r   -18.000000 -26.712888  -9.2871118 0.0000501
B:st-B:r  -63.666667 -72.379555 -54.9537785 0.0000000
S:st-B:r  -61.333333 -70.046222 -52.6204452 0.0000000
B:vl-B:r  -71.666667 -80.379555 -62.9537785 0.0000000
S:vl-B:r  -87.333333 -96.046222 -78.6204452 0.0000000
B:st-S:r  -45.666667 -54.379555 -36.9537785 0.0000000
S:st-S:r  -43.333333 -52.046222 -34.6204452 0.0000000
B:vl-S:r  -53.666667 -62.379555 -44.9537785 0.0000000
S:vl-S:r  -69.333333 -78.046222 -60.6204452 0.0000000
S:st-B:st   2.333333  -6.379555  11.0462215 0.9785200
B:vl-B:st  -8.000000 -16.712888   0.7128882 0.0842128
S:vl-B:st -23.666667 -32.379555 -14.9537785 0.0000015
B:vl-S:st -10.333333 -19.046222  -1.6204452 0.0145554
S:vl-S:st -26.000000 -34.712888 -17.2871118 0.0000004
S:vl-B:vl -15.666667 -24.379555  -6.9537785 0.0002561
```

Easiest way to do Dunnett's is to convert the design to a one-way:

```
agg_comp <- as.factor(paste(agg,comp,sep="_"))
levels(agg_comp)
```

Control: Basalt x static

```
[1] "B_l"  "B_r"  "B_st" "B_vl" "S_l"  "S_r"  "S_st" "S_vl"
```

```
library(DescTools)
DunnettTest(y ~ agg_comp, control = "B_st", conf.level = 0.95)
```

```

  Dunnett's test for comparing several treatments with a control :
    95% family-wise confidence level

$B_st
                diff     lwr.ci      upr.ci    pval
B_l-B_st   32.000000  24.643829  39.3561715 1.5e-11 ***
B_r-B_st   63.666667  56.310495  71.0228381 < 2e-16 ***
B_vl-B_st  -8.000000 -15.356171  -0.6438285  0.0303 *
S_l-B_st   -4.666667 -12.022838   2.6895048  0.3266
S_r-B_st   45.666667  38.310495  53.0228381 4.3e-14 ***
S_st-B_st   2.333333  -5.022838   9.6895048  0.8881
S_vl-B_st -23.666667 -31.022838 -16.3104952 5.0e-08 ***

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Comparing factor level means at each level of other factor

$$
\begin{array}{c|cccc}
 & \multicolumn{4}{c}{b} \\
\hline
 & \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\
a & \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \\
 & \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34}
\end{array}
$$

make pairwise comp.

At each level $j = 1, \dots, b$ of Factor B:

- Tukey's for comparing all pairs among $\bar{\mu}_{ij}, \dots, \bar{\mu}_{aj}$:

$$\bar{Y}_{ij\cdot} - \bar{Y}_{i'j\cdot} \pm q_{\underbrace{a}_{\#\text{ comparisons}},\overbrace{ab(n-1)}^{MS_{Error}\ df},\boxed{\alpha/b}}\, \hat{\sigma} \frac{1}{\sqrt{n}}, \quad 1 \le i < i' \le a.$$

α/b

- Dunnett's for comparing $\bar{\mu}_{ij}, \dots, \bar{\mu}_{a-1,j}$ to baseline μ_{aj}:

$$\bar{Y}_{ij\cdot} - \bar{Y}_{aj\cdot} \pm \overbrace{d_{\underbrace{a,ab(n-1),\alpha/b}_{\text{From table}}}\, \hat{\sigma} \sqrt{\frac{2}{n}}}, \quad i = 1, \dots, a-1.$$

The division of α by b is a Bonferroni correction.

Do likewise when fixing Factor A at a level i.

Tensile strength data (cont)

Compare all pairs of compaction method means when the aggregate type is basalt.

agg = B comp: st l r vl

```
y21. <- mean(y[agg == "B" & comp == "st"])
y22. <- mean(y[agg == "B" & comp == "r"])
y23. <- mean(y[agg == "B" & comp == "l"])
y24. <- mean(y[agg == "B" & comp == "vl"])
alpha <- 0.05
me <- qtukey(1-alpha/a,b,a*b*(n-1)) * sqrt(MSE) / sqrt(n)
ttab <- rbind(c(y21. - y22. - me, y21. - y22. + me),
              c(y21. - y23. - me, y21. - y23. + me),
              c(y21. - y24. - me, y21. - y24. + me),
              c(y22. - y23. - me, y22. - y23. + me),
              c(y22. - y24. - me, y22. - y24. + me),
              c(y23. - y24. - me, y23. - y24. + me))
rownames(ttab) <- c("B:st-r","B:st-l","B:st-vl",
                    "B:r-l","B:r-vl","B:l-vl")
colnames(ttab) <- c("lower","upper")
```

Get the four means

$\hat{\sigma}$

$\{a, ab(n-1), \alpha$

all pairwise differences, $\binom{4}{2} = 6$ of them.

Tukey-adjusted CIs comparing all pairs of compaction methods when the aggregate type is basalt.

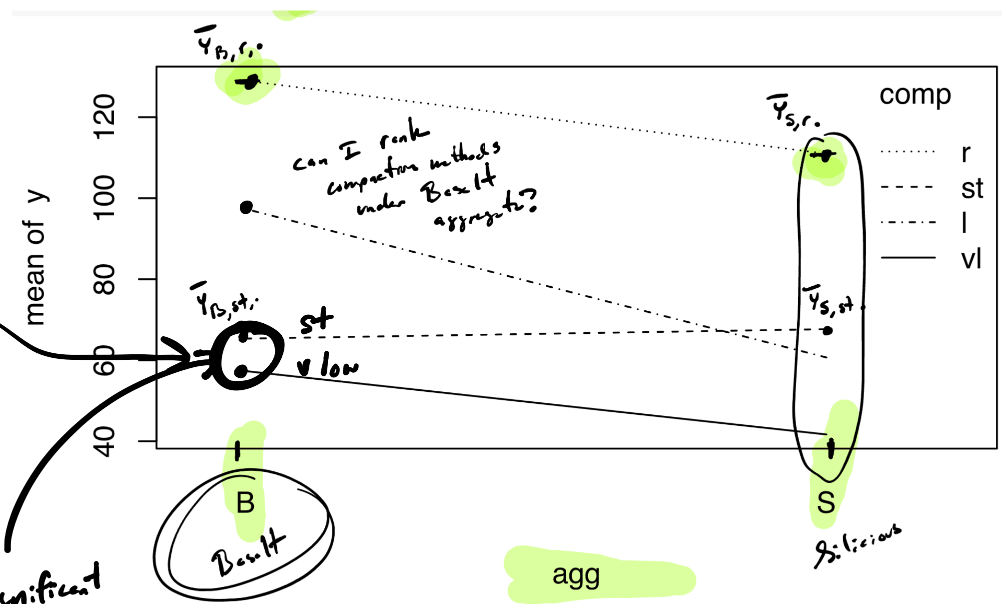
```
ttab
```

differences

```
               lower      upper
B:st-r  -71.75923299 -55.57410
B:st-l  -40.09256633 -23.90743
B:st-vl  -0.09256633  16.09257
B:r-l    23.57410034  39.75923
B:r-vl   63.57410034  79.75923
B:l-vl   31.90743367  48.09257
```

agg

no significant difference.



Tensile strength data (cont)

$a=2$ $\quad n=3 \quad ab(n-1) = 2\cdot4(3-1) = 16$

$b=4$

Compare all compaction method means to the static method when the aggregate type is $\boxed{\text{Basalt.}}$

Use [redacted]

So take [redacted]

$d_{4,16,0.05} = 2.59$

2.59

```
me <-  [redacted] * sqrt(MSE) [redacted] sqrt(2/n).
dtab <- rbind(c(y22. - y21. - me, y22. - y21. + me),
              c(y23. - y21. - me, y23. - y21. + me),
              c(y24. - y21. - me, y24. - y21. + me))
rownames(dtab) <- c("B:r-st","B:l-st","B:vl-st")
colnames(dtab) <- c("lower","upper")
```

Dunnett's

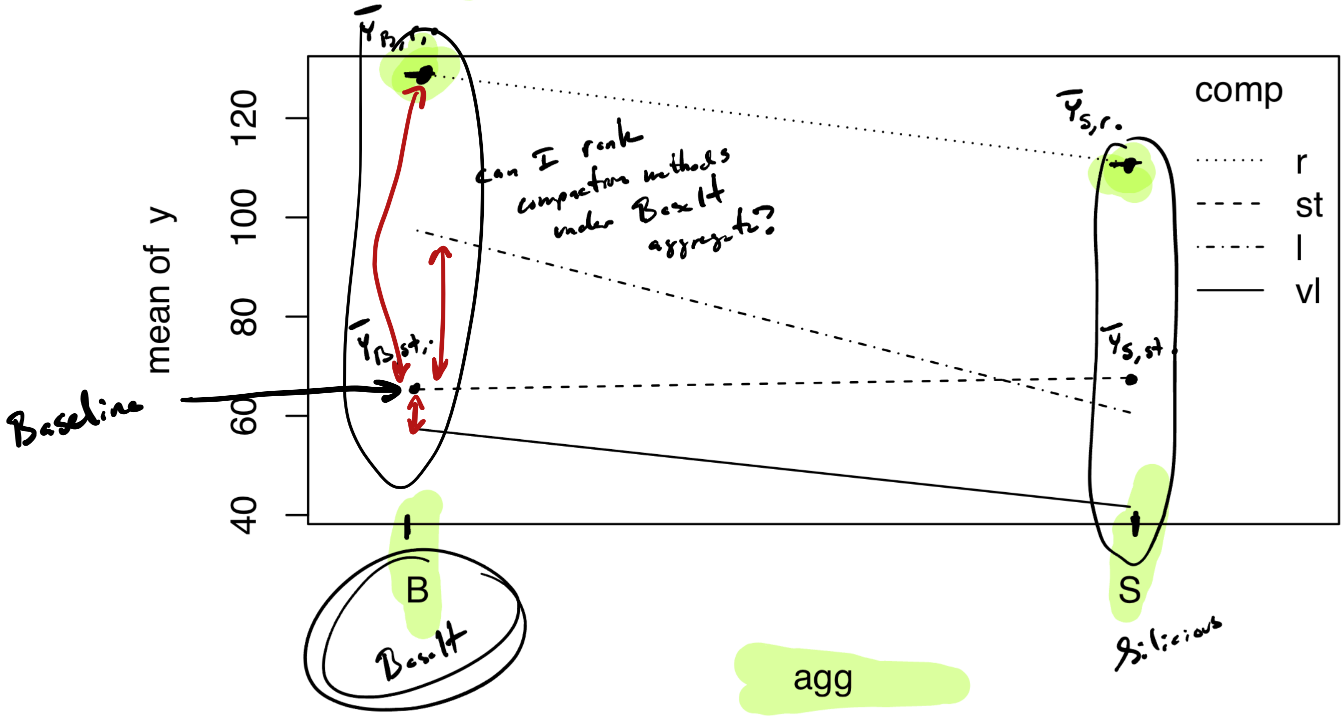

comp
r
st
l
vl
mean of y
120
100
80
60
40
B
S
agg
Baseline
Basalt
Silicious
Can I rank comparison methods under Basalt aggregate?

Table A.5 Critical Values for Dunnett's Two-Sided Test of Treatments versus Control.

Error df	Two-sided α	T = Number of Groups Counting Both Treatments and Control: 2	3	4	5	6	7	8
5	0.05	2.57	3.03	3.29	3.48	3.62	3.73	3.82
5	0.01	4.03	4.63	4.97	5.22	5.41	5.56	5.68
6	0.05	2.45	2.86	3.10	3.26	3.39	3.49	3.57
6	0.01	3.71	4.21	4.51	4.71	4.87	5.00	5.10
7	0.05	2.36	2.75	2.97	3.12	3.24	3.33	3.41
7	0.01	3.50	3.95	4.21	4.39	4.53	4.64	4.74
8	0.05	2.31	2.67	2.88	3.02	3.13	3.22	3.29
8	0.01	3.36	3.77	4.00	4.17	4.29	4.40	4.48
9	0.05	2.26	2.61	2.81	2.95	3.05	3.14	3.20
9	0.01	3.25	3.63	3.85	4.01	4.12	4.22	4.30
10	0.05	2.23	2.57	2.76	2.89	2.99	3.07	3.14
10	0.01	3.17	3.53	3.74	3.88	3.99	4.08	4.16
11	0.05	2.20	2.53	2.72	2.84	2.94	3.02	3.08
11	0.01	3.11	3.45	3.65	3.79	3.89	3.98	4.05
12	0.05	2.18	2.50	2.68	2.81	2.90	2.98	3.04
12	0.01	3.05	3.39	3.58	3.71	3.81	3.89	3.96
13	0.05	2.16	2.48	2.65	2.78	2.87	2.94	3.00
13	0.01	3.01	3.33	3.52	3.65	3.74	3.82	3.89
14	0.05	2.14	2.46	2.63	2.75	2.84	2.91	2.97
14	0.01	2.98	3.29	3.47	3.59	3.69	3.76	3.83
15	0.05	2.13	2.44	2.61	2.73	2.82	2.89	2.95
15	0.01	2.95	3.25	3.42	3.55	3.64	3.71	3.78
16	0.05	2.12	2.42	2.59	2.71	2.80	2.87	2.92
16	0.01	2.92	3.22	3.39	3.51	3.60	3.67	3.73
17	0.05	2.11	2.41	2.58	2.69	2.78	2.85	2.90
17	0.01	2.90	3.19	3.36	3.47	3.56	3.63	3.69
18	0.05	2.10	2.40	2.56	2.68	2.76	2.83	2.89
18	0.01	2.88	3.17	3.33	3.44	3.53	3.60	3.66
19	0.05	2.09	2.39	2.55	2.66	2.75	2.81	2.87
19	0.01	2.86	3.15	3.31	3.42	3.50	3.57	3.63
20	0.05	2.09	2.38	2.54	2.65	2.73	2.80	2.86
20	0.01	2.85	3.13	3.29	3.40	3.48	3.55	3.60
25	0.05	2.06	2.34	2.50	2.61	2.69	2.75	2.81
25	0.01	2.79	3.06	3.21	3.31	3.39	3.45	3.51
30	0.05	2.04	2.32	2.47	2.58	2.66	2.72	2.77
30	0.01	2.75	3.01	3.15	3.25	3.33	3.39	3.44
40	0.05	2.02	2.29	2.44	2.54	2.62	2.68	2.73
40	0.01	2.70	2.95	3.09	3.19	3.26	3.32	3.37
60	0.05	2.00	2.27	2.41	2.51	2.58	2.64	2.69
60	0.01	2.66	2.90	3.03	3.12	3.19	3.25	3.29

This table produced from the SAS System using function PROBMC('DUNNETT2',,1 − α,df,k), where $k = T - 1$.

within the Basalt level of the agg factor, we use 4

Figure 1: Table A.5 from Mohr, Wilson, and Freund (2021)

Dunnett's comparison of compaction method means to the static method when the aggregate type is basalt:

```
dtab
```

```
             lower     upper
B:l-st   25.48198 38.518024
B:r-st   57.14864 70.184690
B:vl-st -14.51802 -1.481976
```

Correct

Interaction *not* significant. Then what?

If the interaction is not significant:

1. We can interpret main effects.
2. We can make meaningful comparisons among marginal means.

μ_{11}	μ_{12}	μ_{13}	μ_{14}	$\bar{\mu}_{1\cdot}$
μ_{21}	μ_{22}	μ_{23}	μ_{24}	$\bar{\mu}_{2\cdot}$
μ_{31}	μ_{32}	μ_{33}	μ_{34}	$\bar{\mu}_{3\cdot}$
$\bar{\mu}_{\cdot 1}$	$\bar{\mu}_{\cdot 2}$	$\bar{\mu}_{\cdot 3}$	$\bar{\mu}_{\cdot 4}$	$\bar{\mu}_{\cdot\cdot}$

compare marginal means for factor A

If no interaction, then we can focus on the margins of the table.

marginal mean for factor B.

Comparing marginal means in the absence of interaction

For making comparisons among the marginal means across Factor A:

- Tukey's for comparing all pairs among $\bar{\mu}_{1.}, \dots, \bar{\mu}_{a.}$:

$$\bar{Y}_{i..} - \bar{Y}_{i'..} \pm q_{a,ab(n-1),\alpha}\, \hat{\sigma} \frac{1}{\sqrt{nb}}, \quad 1 \le i < i' \le a.$$

- Dunnett's for comparing $\bar{\mu}_{1.}, \dots, \bar{\mu}_{a-1.}$ to $\bar{\mu}_{a.}$:

$$\bar{Y}_{i..} - \bar{Y}_{a..} \pm d_{a,ab(n-1),\alpha}\, \hat{\sigma} \sqrt{\frac{2}{nb}}, \quad i = 1, \dots, a-1.$$

Still use $\hat{\sigma} = \sqrt{\mathrm{MS}_{\mathrm{Error}}}$.

Do likewise for making comparisons among $\bar{\mu}_{.1}, \dots, \bar{\mu}_{.b}$ of Factor B.

Serum glucose example from Kuehl (2000)

Two methods for measuring serum glucose level at three glucose levels.

$n = 3$

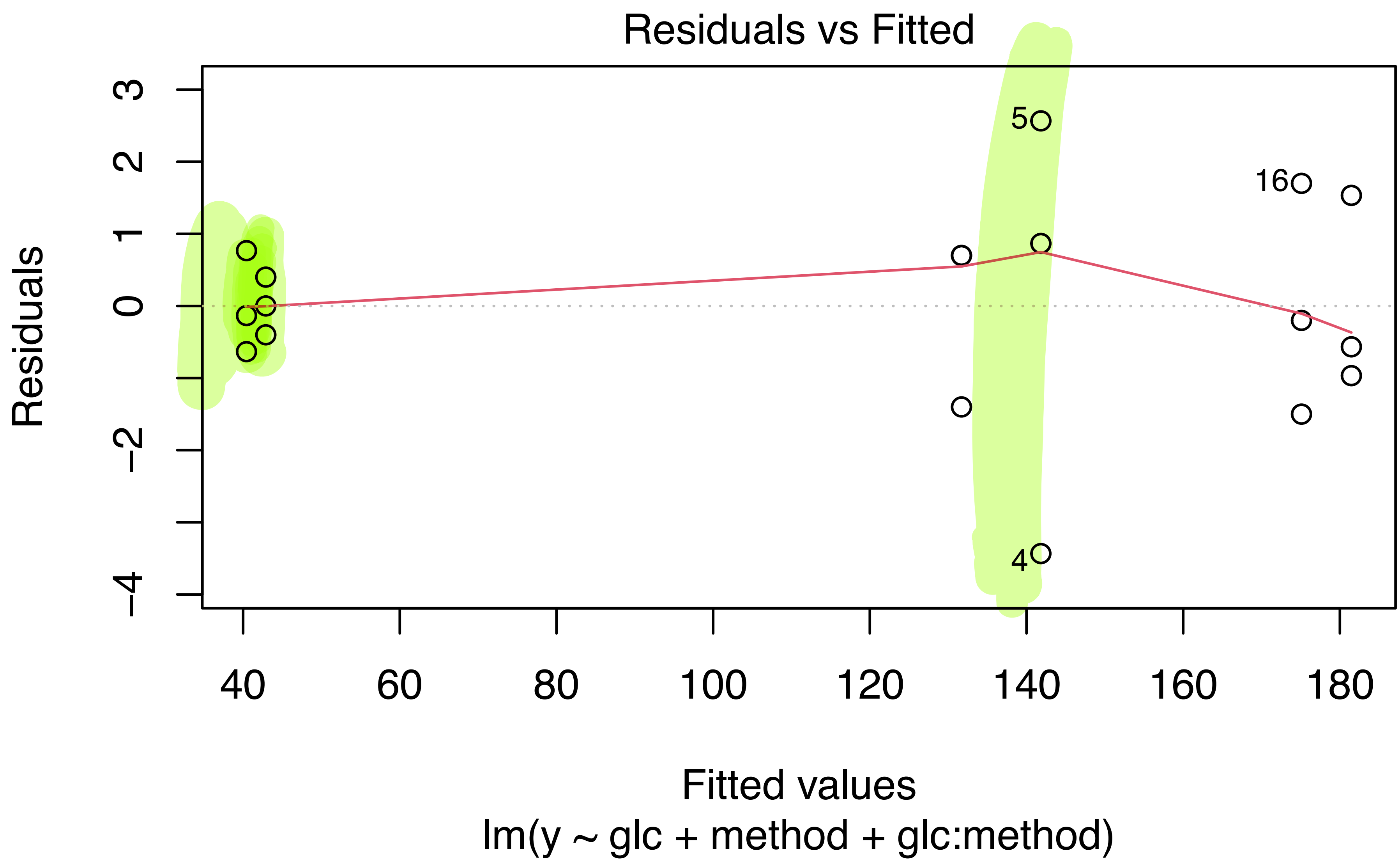
	Method 1			Method 2		
Glucose Level	*1*	*2*	*3*	*1*	*2*	*3*
	42.5	138.4	180.9	39.8	132.4	176.8
	43.3	144.4	180.5	40.3	132.4	173.6
	42.9	142.7	183.0	41.2	130.3	174.9

Source: Dr. J. Anderson, Beckman Instruments Inc.

```
y <- c(42.5,43.3,42.9,138.4,144.4,142.7,180.9,180.5,183.0,
       39.8,40.3,41.2,132.4,132.4,130.3,176.8,173.6,174.9)
glc <- as.factor(c(1,1,1,2,2,2,3,3,3,1,1,1,2,2,2,3,3,3))
method <- as.factor(c(1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2))
```

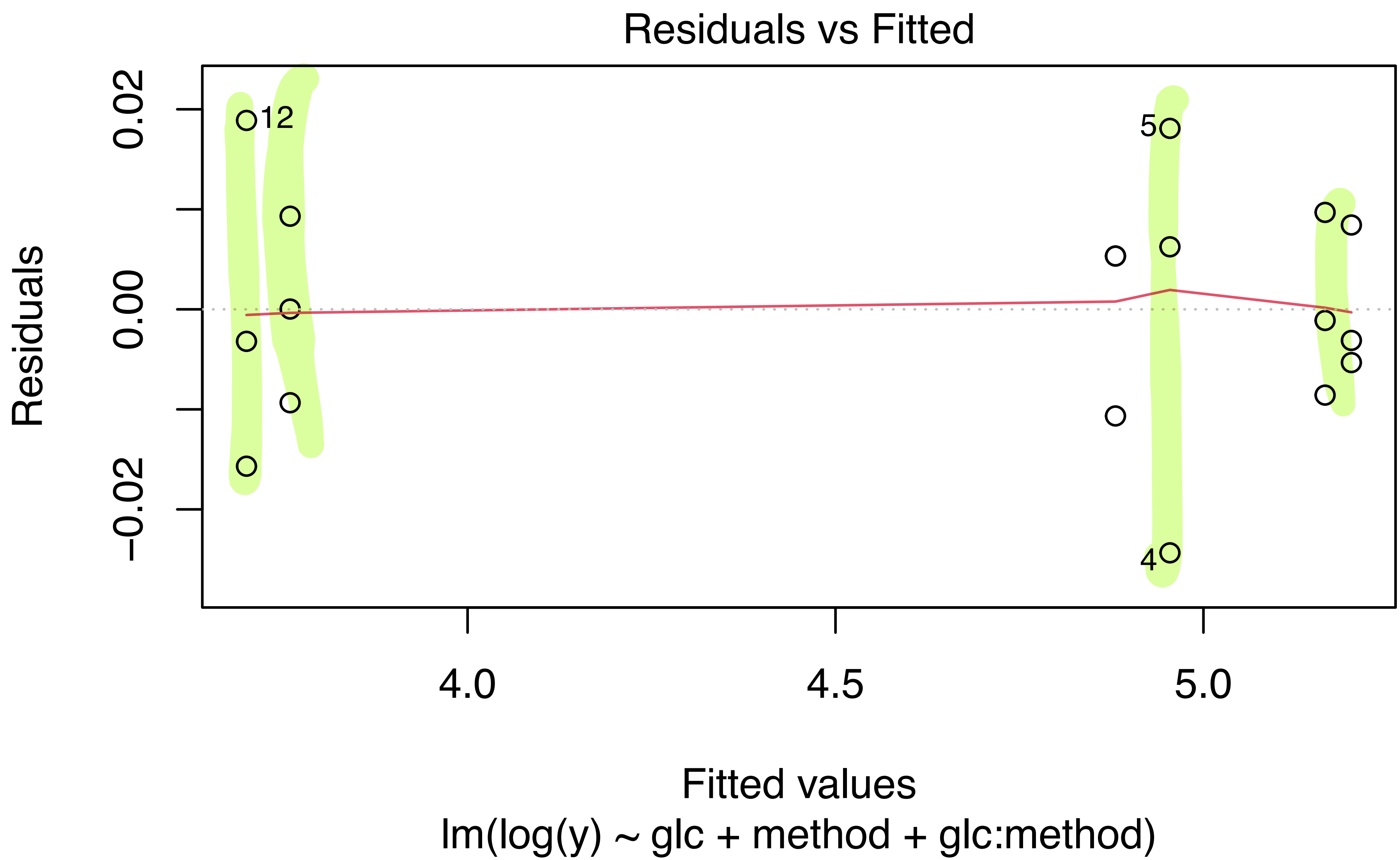
- Is there an interaction between the method and the glucose level?
- If not, can we describe the main effect of the method?

```
lm_glc <- lm(y ~ glc + method + glc:method)
plot(lm_glc,which = 1)
```



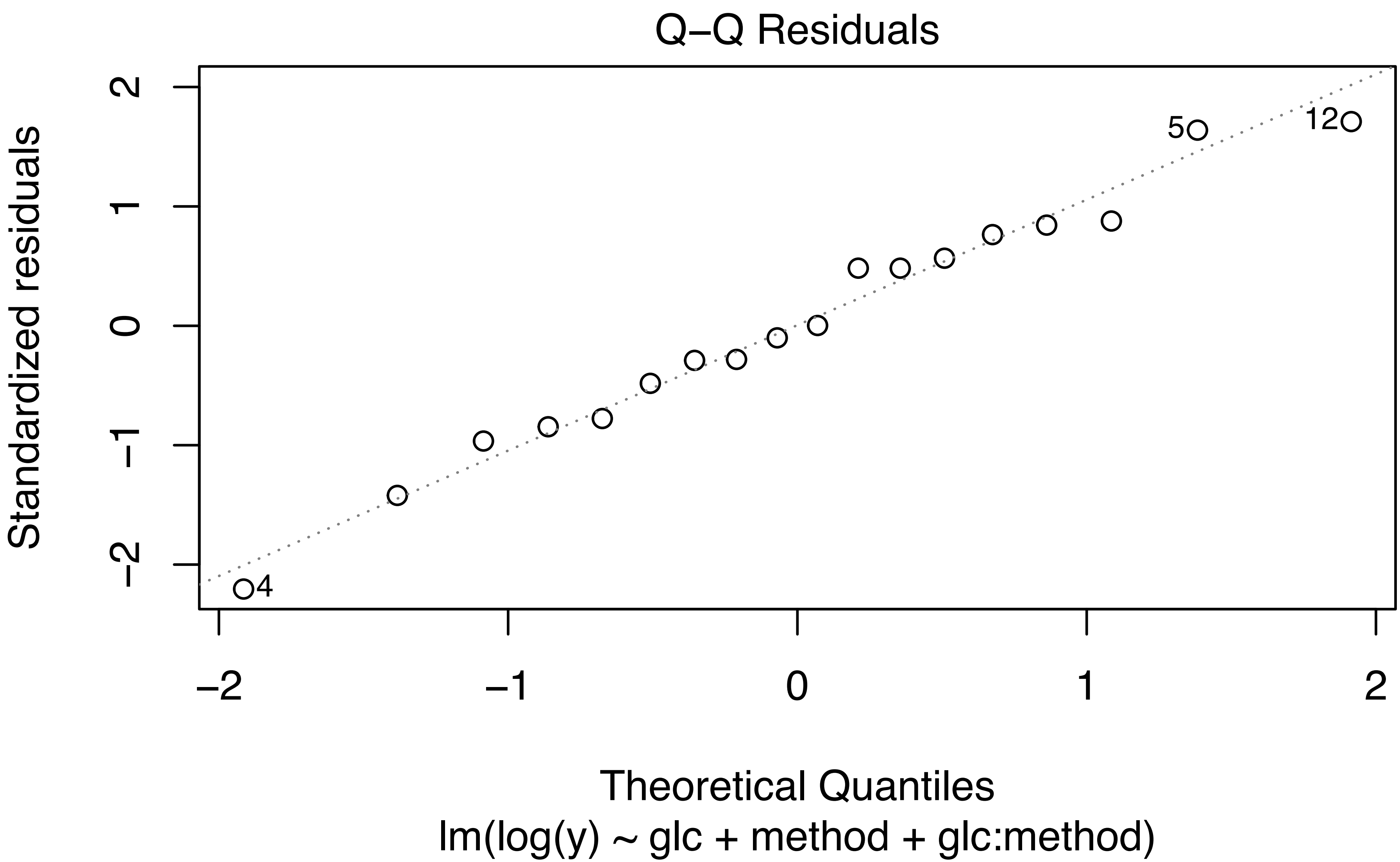
Variance appears smaller at lower glucose level. Try using $\log(Y_{ijk})$.

```r
lm2_glc <- lm(log(y) ~ glc + method + glc:method)
plot(lm2_glc,which = 1)
```



This looks better.

```
plot(lm2_glc,which = 2)
```



Normality check looks okay.

A = Method $a = 2$ $n = 3$

B = Glucose level $b = 3$

```r
anova(lm2_glc)
```

$a-1 = 2-1 = 1$

$b-1 = 3-1 = 2$

$(a-1)(b-1) = (2-1)(3-1) = 2$

```
Analysis of Variance Table

Response: log(y)
           Df Sum Sq Mean Sq    F value    Pr(>F)
method      1 0.0143  0.0143    78.1091 1.337e-06 ***
glc         2 7.1935  3.5967 19670.4837 < 2.2e-16 ***
method:glc  2 0.0011  0.0006     3.0574    0.0845 .
Residuals  12 0.0022  0.0002
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

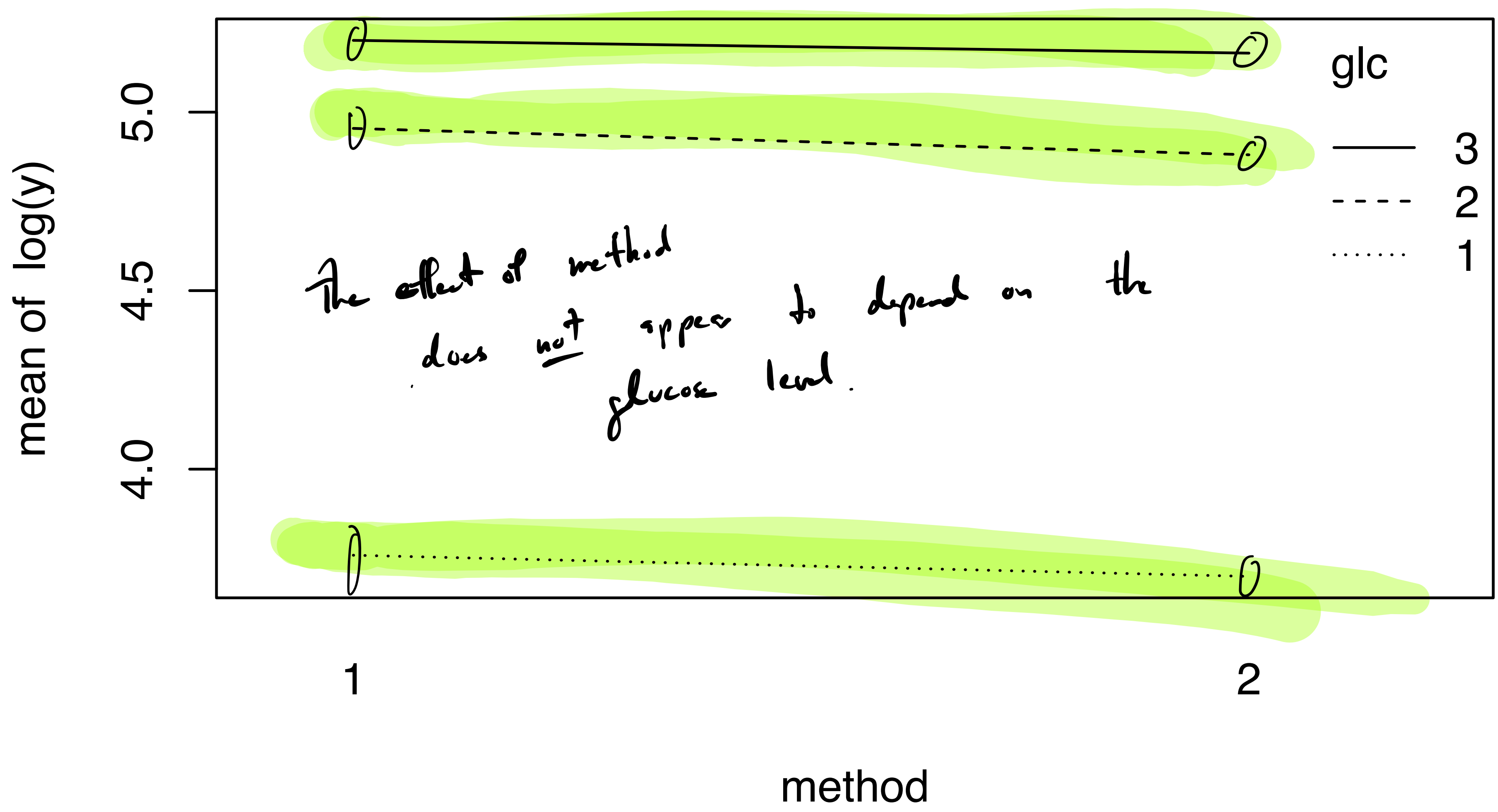
Fail to reject H_0: No interaction

$ab(n-1) = 2 \cdot 3 \cdot (3-1) = 12$

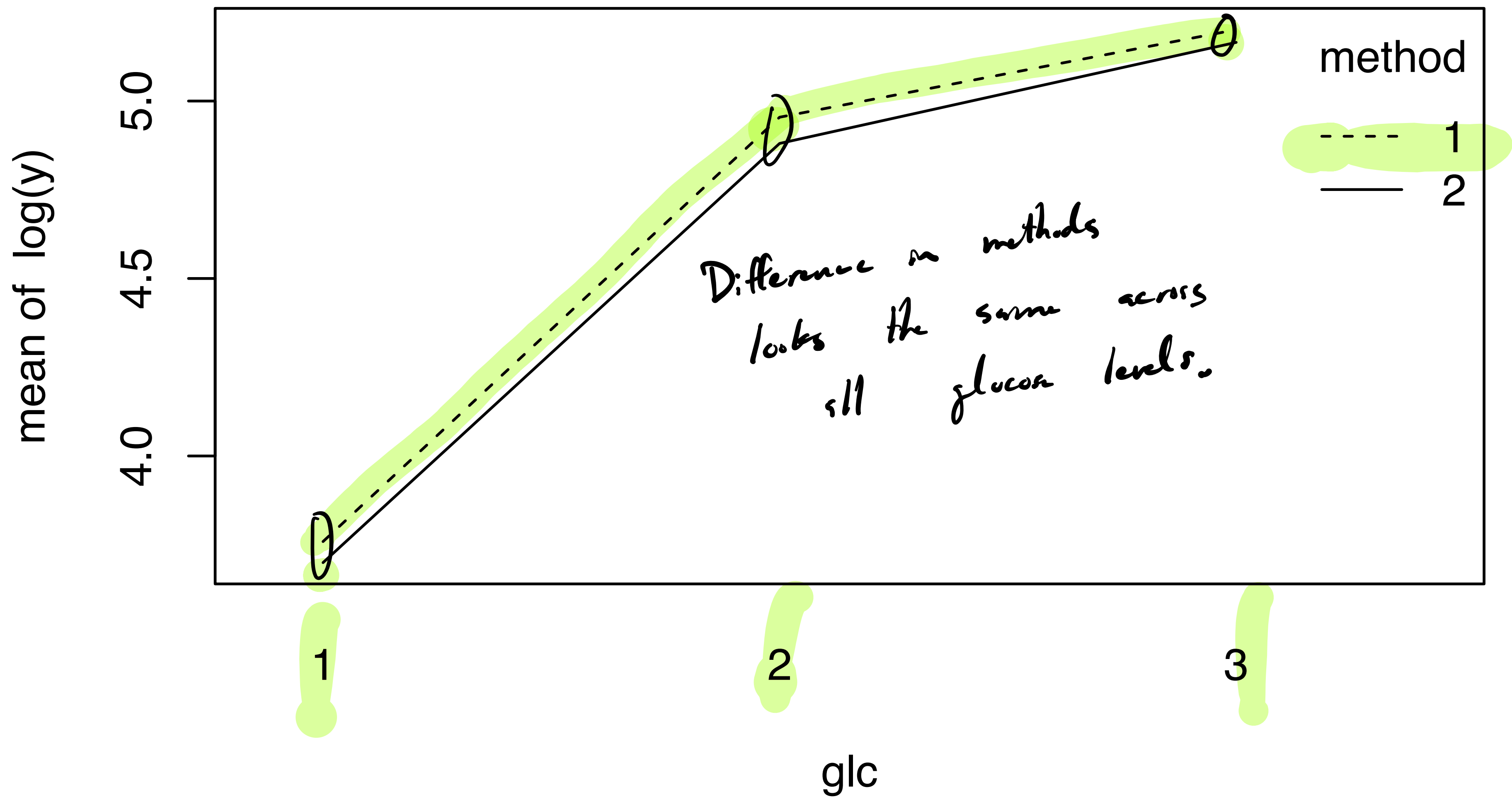
There is only weak evidence of interaction. Check interaction plot.

$MS_{Error} = 0.0002$

```
interaction.plot(method,glc,log(y))
```



```r
interaction.plot(glc,method,log(y))
```



It appears safe to ignore the interaction and report on main effects.

Glucose

Method				
	μ_{11}	μ_{12}	μ_{13}	$\bar\mu_{1\cdot}$
	μ_{21}	μ_{22}	μ_{23}	$\bar\mu_{2\cdot}$
	$\bar\mu_{\cdot 1}$	$\bar\mu_{\cdot 2}$	$\bar\mu_{\cdot 3}$	

Build C.I. for $\bar\mu_{1\cdot} - \bar\mu_{2\cdot}$, diff in marginal method means.

Glucose

Method				
	$\bar Y_{11\cdot}$	$\bar Y_{12\cdot}$	$\bar Y_{13\cdot}$	$\bar Y_{1\cdot\cdot}$
	$\bar Y_{21\cdot}$	$\bar Y_{22\cdot}$	$\bar Y_{23\cdot}$	$\bar Y_{2\cdot\cdot}$
	$\bar Y_{\cdot 1\cdot}$	$\bar Y_{\cdot 2\cdot}$	$\bar Y_{\cdot 3\cdot}$	

$a=2$, $b=3$, $n=3$, $\alpha=0.05$ $\qquad \hat\sigma = \sqrt{MS_{Error}} = \sqrt{0.0002}$

$$\bar Y_{i\cdot\cdot} - \bar Y_{i'\cdot\cdot} \pm \underbrace{q_{a,\overbrace{ab(n-1)}^{12},\alpha}} \hat\sigma \frac{1}{\sqrt{nb}},$$

$q_{2,12,0.05}$

$= \text{qtukey}(0.95,\ \underset{\#\text{ means}}{2},\ \underset{df\ MS_{Error}}{12}) = 3.081$

Call the method Factor A; build a CI for $\bar{\mu}_{1.} - \bar{\mu}_{2.}$ (just one comparison).

Since $a = 2$, $b = 3$, and $n = 3$, use $\boxed{\bar{Y}_{1..} - \bar{Y}_{2..} \pm q_{2,2\cdot 3(3-1),0.05} \; \hat{\sigma} \frac{1}{\sqrt{3 \cdot 3}}}$.

```
a <- 2
b <- 3
n <- 3
alpha <- 0.05
y1.. <- mean(log(y[method == 1])) # remember we are using log(y)
y2.. <- mean(log(y[method == 2]))
MSE <- sum(lm2_glc$residuals^2) / (a*b*(n-1))
me <- qtukey(1-alpha,a,a*b*(n-1)) * sqrt(MSE) / sqrt(n*b)
lo <- y1.. - y2.. - me
up <- y1.. - y2.. + me
c(lo,up)
```

SSE / (ab(n-1))

C.I. for difference in Method 1 and Method 2 measurements (1 - 2).

```
[1] 0.04244809 0.07022545
```

= The measurements (in log scale) from Method 1 are on average greater than those from Method by an amount in the interval (0.042, 0.070)

Since $a = 2$, $q_{a,ab(n-1),\alpha} = \sqrt{2} \cdot t_{ab(n-1),\alpha/2}$, so it is just a t-interval.

$q_{2,12,0.05} = 3.081 = \sqrt{2} \cdot t_{12,0.025}$

$\parallel$

qt(.975, 12) = 2.1788

Possible workflow for factorial experiments

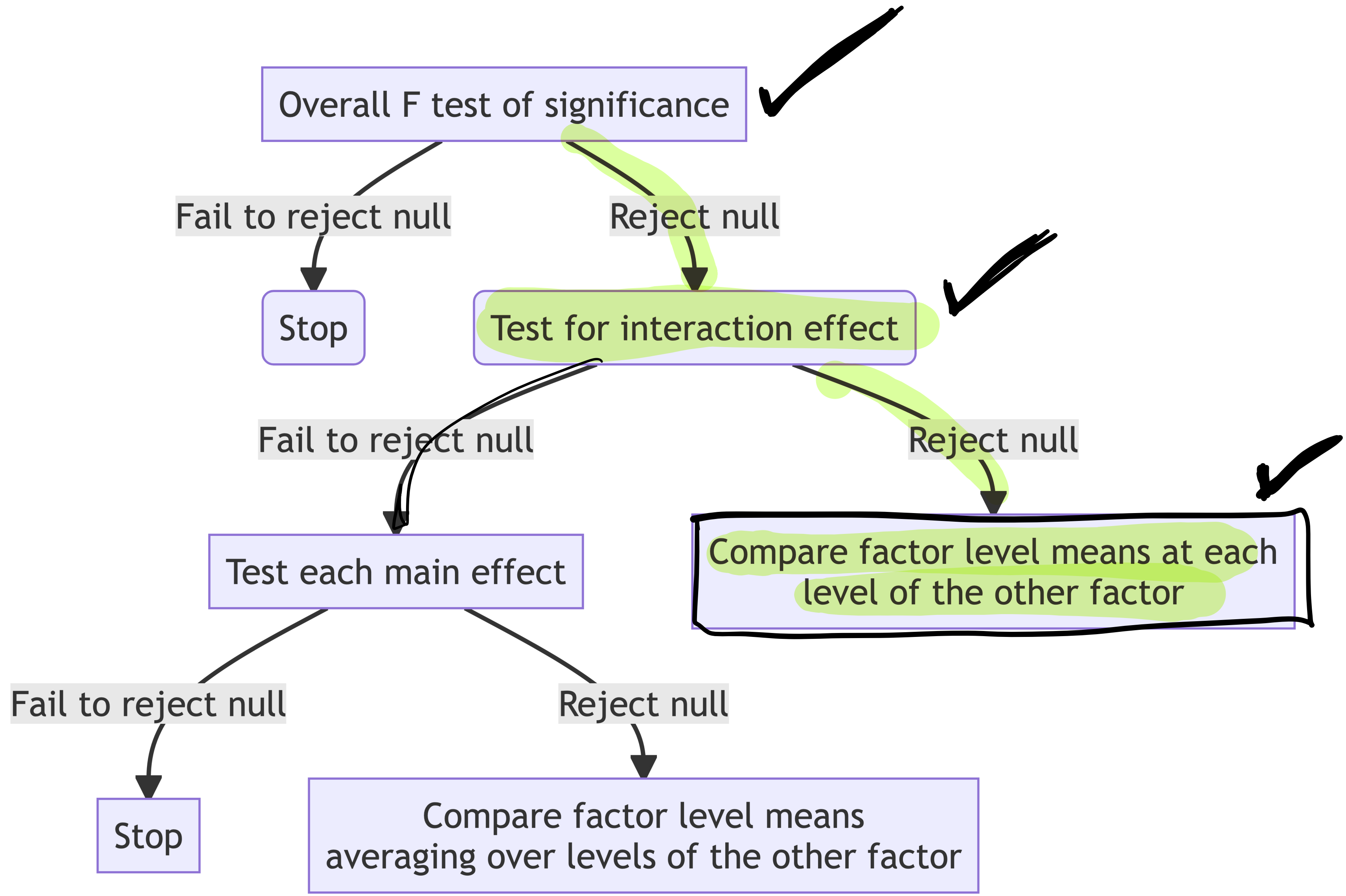


Figure 2: A justifiable workflow for analyzing factorial experiment data.

References

"2" comes from the fact that

$$Var(\bar{Y}_{1..} - \bar{Y}_{2..}) = \frac{2\sigma^2}{nb}$$

$$\bar{Y}_{1..} - \bar{Y}_{2..} \pm t_{ab(n-1),\alpha/2} \sqrt{MS_{Error}} \sqrt{\frac{2}{nb}}$$

$$\bar{Y}_{1..} - \bar{Y}_{2..} \pm q_{a,ab(n-1),\alpha} \sqrt{MS_{Error}} \sqrt{\frac{1}{nb}}$$

Kuehl, R. O. 2000. *Design of Experiments: Statistical Principles of Research Design and Analysis*. Duxbury/Thomson Learning.

Mohr, Donna L, William J Wilson, and Rudolf J Freund. 2021. *Statistical Methods*. Academic Press.