STAT 712 fa 2022 Lec 2 slides

Counting rules

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Motivation to study counting rules

If all outcomes in a finite sample space Ω are equally likely, we have

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } \Omega\}} \quad \text{for all } A \subset \Omega.$$

Example: The probability space $(\Omega, \mathcal{B}, P)$ for rolling two dice is given by

A: {sum of rolls is 7}

$p(A) = \frac{6}{36} = \frac{1}{6}$

$$\Omega = \left\{ \begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

$$\mathcal{B} = \{\text{all subsets of } \Omega, \text{ including } \Omega \text{ itself}\}$$

$$P(A) = \frac{\#\{\text{points in } A\}}{36} \quad \text{for all } A \in \mathcal{B}.$$

Such situations lead us to study counting rules. . .

Fundamental theorem of counting

If a job consists of K tasks such that the tasks may be completed in $n_1, \dots, n_K$ ways, respectively, then there are

$$\prod_{k=1}^{K} n_k = n_1 \times n_2 \times \cdots \times n_K$$

ways to do the job.

Exercise: Consider a birthday party with 10 children. In how many ways can

1. the children line up to take turns hitting a piñata?
2. 3 of the 15 parents present be chosen to supervise?
3. the children hit or miss on the first 10 swings (sequences of hits and misses)?
4. the 50 pieces of identical candy be distributed amongst the 10 children.

(1)

tasks	# ways
choose 1st child	10
2nd	9
3rd	8
⋮	
10th	1

\# ways to do job:

$10\cdot 9\cdot 8\cdot 7\cdot \ldots \cdot 2\cdot 1$

$= 10!$

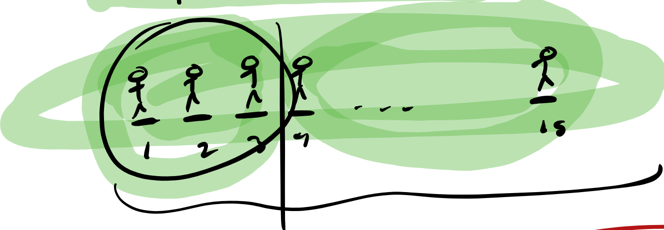
"factorial"

$\boxed{\text{Ordered without repl.} \quad N=10,\ r=10 \quad \frac{N!}{(N-r)!} = \frac{10!}{(10-10)!} = \frac{10!}{\underbrace{0!}_{=1}} = 10!}$

(2) Choose 3 of 15 parents

$$\binom{15}{3} = \frac{15!}{3!(15-3)!}.$$

Consider lining up all 15 parents in order:



15! ways

Break into these tasks

task	# ways
choose 3 of 15 parents to stand in front of line	?
order front 3	3!
order remaining 15 - 3	(15 - 3)!

Unordered without repl.

$N = 15$

$r = 3$

$$\binom{N}{r} = \binom{15}{3} = \frac{N!}{r!(N-r)!}$$

$$(?)\cdot 3!\cdot(15-3)! = 15! \iff (?) = \frac{15!}{3!(15-3)!} = \binom{15}{3}$$

15 "choose" 3

(3)

tasks	# ways
1st swing	2 (hit/miss)
2nd	2
3rd	2
10th	2

ways to do job: 2^{10}

Ordered with repl.

$N = 2$

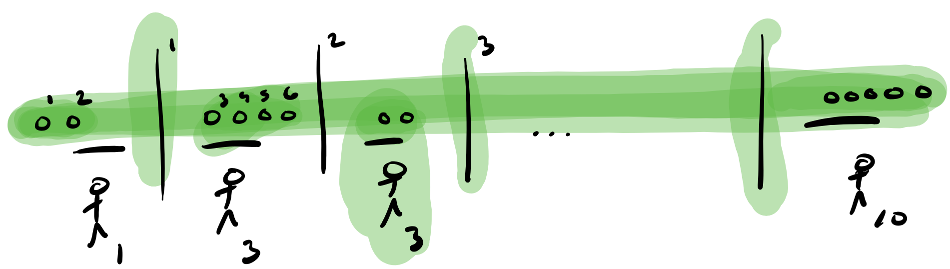
$r = 10$

$N^r = 2^{10}$

(4)

50 candies

10 kids → 9 bin walls



$$\# \text{ ways} = \frac{(50+9)!}{50!\ 9!}$$

Unordered with replacement

$N = 10$

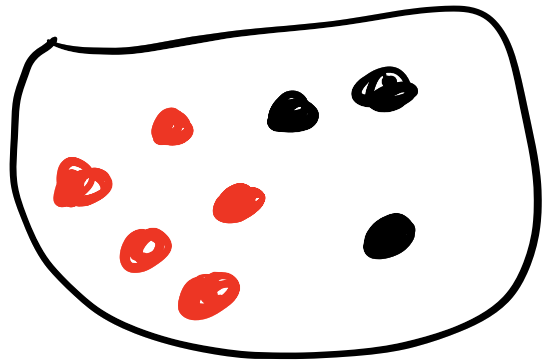
$r = 50$

$$\binom{N+r-1}{r} = \binom{50+10-1}{50} = \frac{(50+9)!}{50!\,(50+9-50)!} = \frac{(50+9)!}{50!\ 9!}.$$

Ways to draw r things from N things

	Without replacement	With replacement
Ordered	$\dfrac{N!}{(N-r)!}$	N^r
Unordered	$\dbinom{N}{r}$	$\dbinom{N+r-1}{r}$

Discuss: Heuristics for above formulas.



$N = 8$ # marbles

$M = 5$ # red

$K =$ # drawn

$N - M =$ # non-red

[Hypergeometric distribution]

Exercise: Consider drawing K marbles from a bag containing N marbles, M of which are red. Give the probability of

1. drawing K red marbles (assuming $M \geq K$). [At least K red marbles in bag]
2. drawing 0 red marbles (assuming $N - M \geq K$). [At least K non-red marbles]
3. drawing x red marbles for $x \in \{\max(K - (N - M), 0), \dots, \min(M, K)\}$

(1) $P(\text{Draw } K \text{ red}) = \dfrac{\#\{\text{ways to draw } K \text{ red marbles from bag}\}}{\#\{\text{ways to draw } K \text{ marbles from bag of } N\}}$

$$= \frac{\binom{M}{K}}{\binom{N}{K}}$$

② $P(\text{Draw } 0 \text{ red}) = \dfrac{\#\{\text{Ways to draw } 0 \text{ red marbles}\}}{\#\{\text{ways to draw } K \text{ marbles from bag of } N\}}$

$$= \frac{\binom{N-M}{K}}{\binom{N}{K}}$$

③ $$P(\text{Draw } x \text{ red marbles}) = \frac{\overset{\text{task 1}}{\overset{\downarrow}{\binom{M}{x}}} \cdot \overset{\text{task 2}}{\overset{\downarrow}{\binom{N-M}{K-x}}}}{\binom{N}{K}}$$

Number of ways to partition

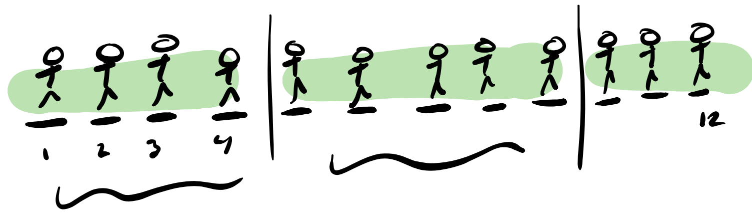
The number of ways to partition N things into K groups of sizes $n_1, \dots, n_K$, where $n_1 + \cdots + n_K = N$, is

$$\frac{N!}{n_1! \times \cdots \times n_K!}.$$

Exercise: Suppose 12 passengers are assigned at random to ride in three vehicles taking 4, 5, and 3 passengers, respectively.

1. In how many ways can the passengers be assigned to the different vehicles?
2. Suppose you and a friend are among the passengers. What is the probability that you will ride in a vehicle with your friend?

(1)



Task: order all 12 in $12!$ ways.

Decompose:

tasks	# ways
partition into groups of 4, 5, and 3	?
order among first 4	$4!$
order among next 5	$5!$
order among last 3	$3!$

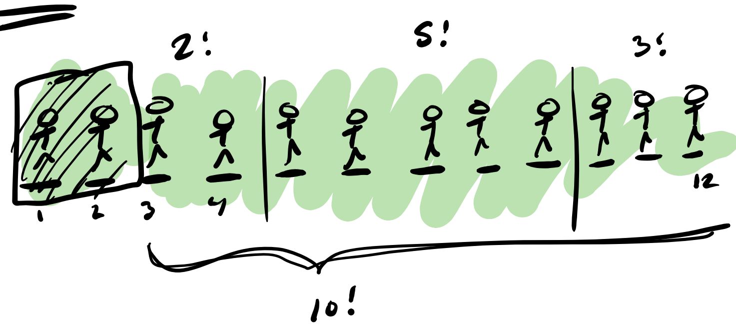
$$(?) \cdot 4! \cdot 5! \cdot 3! = 12!$$

$$(?) = \frac{12!}{4!\,5!\,3!}$$

(2) $P(\text{You ride with friend}) = \dfrac{\#\{\text{ways you are with friend}\}}{\#\{\text{ways passengers grouped in vehicles}\}}$

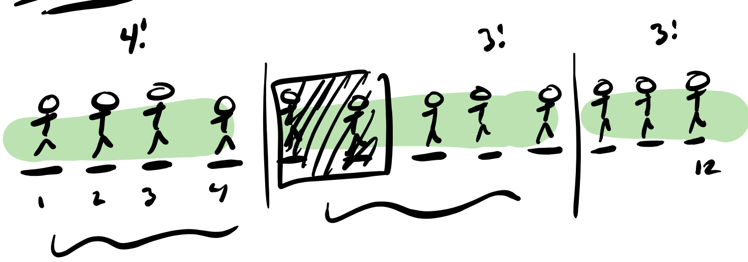
$\#\{\text{ways you are with friend}\}$ Consider possibilities:

1st car



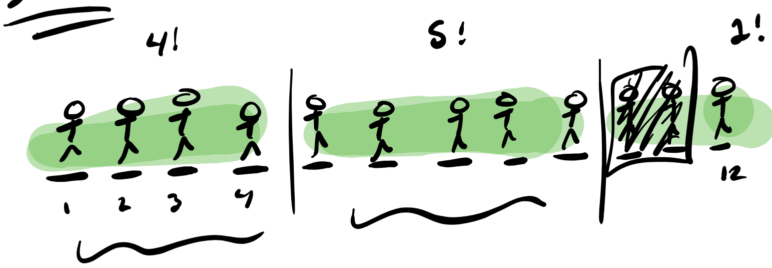
$\#$ ways $\dfrac{10!}{2!\,5!\,3!}$

2nd car:



$\#$ ways $\dfrac{10!}{4!\,3!\,3!}$

3rd car



$\#$ ways $\dfrac{10!}{4!\,5!\,1!}$

$$P(\text{You ride with friend}) = \frac{\dfrac{10!}{3!\,5!\,3!} + \dfrac{10!}{4!\,3!\,3!} + \dfrac{10!}{4!\,5!\,1!}}{\dfrac{12!}{4!\,5!\,3!}} = .2879$$

More fun practice...

\# five card hands: $\binom{52}{5}$

Exercise: Consider drawing 5 cards from a 52-card deck. Find the number of

1. hands with the ace of diamonds.
2. full house hands (three cards of one rank and two of another).
3. flush hands (5 cards of the same suit but not sequential).
4. straight hands (5 cards in sequence but not all of the same suit).

Then compute the corresponding probabilities.

1

tasks	# ways
grab A♢	1
draw other four cards	$\binom{51}{4}$

\# ways $\binom{51}{4} = 249{,}900$

$$P(A\diamondsuit) = \frac{\binom{51}{4}}{\binom{52}{5}} = \frac{\frac{51!}{4!\,47!}}{\frac{52!}{5!\,47!}} = \frac{5}{52}.$$

②

tasks	# ways
choose a rank in which to have 3 cards	13
choose rank in which to have 2 cards	12
Draw 3 cards of first rank	$\binom{4}{3}$
Draw 2 cards of 2nd rank	$\binom{4}{2}$

A - K

$$\# \text{ways} = 13 \cdot 12 \cdot \underline{\binom{4}{3} \cdot \binom{4}{2}}$$

$$= 13 \cdot 12 \cdot 4 \cdot 6$$

$$= 13 \cdot 12 \cdot 24$$

$$= 3{,}744$$

$$P(\text{Full house}) = \frac{3{,}744}{\binom{52}{5}}$$

③ Flush

tasks	# ways
choose suit	4
draw 5 cards from the suit, but not sequential	$\binom{13}{5} - 10$

A - 5

⋮

10 J Q K A

} 10 straight flushes

$$\# \text{ways} = 4\left[\binom{13}{5} - 10\right] = 5{,}108.$$

$$P(\text{flush}) = \frac{5{,}108}{\binom{52}{5}}.$$