STAT 712 fa 2022 Lec 3 slides

Conditional probability and independence

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Conditional probability

Given a probability space $(\Omega, \mathcal{B}, P)$ and a set $B \in \mathcal{B}$ such that $P(B) > 0$ the *conditional probability* of A given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{for all } A \in \mathcal{B}.$$

Exercise: Show that $P(\cdot|B)$ satisfies the Kolmogorov axioms.

Exercise: Consider the probability space $(\Omega, \mathcal{B}, P)$ for rolling two dice:

$$\Omega = \left\{ \begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

$$\mathcal{B} = \{\text{all subsets of } \Omega, \text{ including } \Omega \text{ itself}\}$$

$$P(A) = \#\{\text{points in } A\}/36 \quad \text{for all } A \in \mathcal{B}.$$

Let $A = \{\text{roll doubles}\}$, $B = \{\text{absolute difference in rolls less than 2}\}$, and $C = \{\text{sum of rolls 10 or more}\}$ and find

1. $P(B)$
2. $P(A|B)$
3. $P(A \cup B|C)$

Bayes' Rule

Let $A_1, A_2, \cdots \in \mathcal{B}$ be a partition of the sample space and let $B \in \mathcal{B}$ be any set with $P(B) > 0$. Then

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)P(A_j)}.$$

This is called *Bayes' rule*.

Note $P(B) = \sum_{j=1}^{\infty} P(B|A_j)P(A_j) = \sum_{j=1}^{\infty} P(B \cap A_j)$ is the *law of total prob*.

Baby Bayes'

If A and B are events in $\mathcal{B}$ and $P(B) > 0$, then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Exercise: Consider an imperfect test for the presence of an infection such that

$$\begin{aligned} P(T|I) &= 0.92 \quad \text{(Sensitivity)} \\ P(T^c|I^c) &= 0.98 \quad \text{(Specificity)} \end{aligned}$$

Moreover, suppose the infection is present in 5% of the population.

For an individual drawn at random from the population, find

1. $P(I|T)$
2. $P(I^c|T^c)$

Exercise: Suppose there are K bags, each containing K marbles, and that k of the marbles in bag k are red, for $k = 1, \dots, K$. Select a bag at random and draw a marble from it:

1. What is the probability of drawing a red marble?
2. Given you drew a red marble, what is the probability you drew from bag k?

Independence

Two events A and B in $\mathcal{B}$ are called *independent* if

$$P(A \cap B) = P(A)P(B).$$

Also: If A, B independent, so are the pairs of events A, B^c and A^c, B^c and A^c, B^c.

Equivalent definitions of independence

The following statements are equivalent:

1. $P(A \cap B) = P(A)P(B)$
2. $P(A) = P(A|B)$
3. $P(B) = P(B|A)$

Exercise: Given a probability space $(\Omega, \mathcal{B}, P)$ and events $A, B \in \mathcal{B}$, show:

1. If $P(A \cap B) = 0$ and $P(A) = 1$, then $P(B) = 0$.
2. If $P(A) > 0$, $P(B) > 0$, and $P(A) < P(A|B)$, then $P(B) < P(B|A)$.
3. If A and B are independent, then A^c and B^c are independent.

Exercise: Let $\Omega = \{1, 2, \ldots, n\}$ and A and B be (independently) randomly selected from the collection of all subsets of Ω. Show that $P(A \subset B) = (3/4)^n$.

Multiplication rule

For any $A_1, A_2, \ldots, A_n \in \mathcal{B}$, we have

$$P\left(\cap_{i=1}^{n} A_i\right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_n| \cap_{i=1}^{n-1} A_i).$$

Exercise: Prove the result by induction.

Exercise: Find prob. of hand 10♣, J♣, Q♣, K♣, A♣ from a 52-card deck.

Mutual independence

A collection of events $A_1, A_2, \dots, A_n$ are

1. *mutually independent* if for any subcollection $A_{i_1}, \dots, A_{i_K}$, we have

$$P\left(\bigcap_{j=1}^{K} A_{i_j}\right) = \prod_{j=1}^{K} P(A_{i_j})$$

2. *pairwise independent* if $P(A_j \cap A_i) = P(A_j)P(A_i)$ for all $i \neq j$.

Extend independence between two events to independence among ≥ 2 events.

Note: Can have pairwise indep. without mutual independence (ex 1.3.11 of CB).

Exercise: Find the probability of rolling 3 sixes in 8 rolls of a die.