STAT 712 fa 2022 Lec 3 slides

Conditional probability and independence

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

$P(A) \leftarrow$ "unconditional probability"

Conditional probability

Given a probability space $(\Omega, \mathcal{B}, P)$ and a set $B \in \mathcal{B}$ such that $P(B) > 0$ the *conditional probability* of A given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{for all } A \in \mathcal{B}.$$

Probability that A happens, given that B happens.

Exercise: Show that $P(\cdot|B)$ satisfies the Kolmogorov axioms.

(i) $P(A|B) \overset{?}{\geq} 0 \quad \forall \; A \in \mathcal{B}$

$$P(A|B) = \frac{\overset{\geq 0}{P(A \cap B)}}{\underbrace{P(B)}_{> 0}} \geq 0 \quad \checkmark$$

(ii) $P(\Omega \mid B) \overset{?}{=} 1$

$$P(\Omega \mid B) = \frac{P(\overbrace{\Omega \cap B}^{B})}{P(B)} = \frac{P(B)}{P(B)} = 1 \quad \checkmark$$

(iii) $A_1, A_2, \ldots \in \mathcal{B}$ pwise disjoint

$$P\left(\bigcup_{n=1}^{\infty} A_n \mid B\right) \overset{?}{=} \sum_{n=1}^{\infty} P(A_n \mid B) \quad \text{[countable additivity]}$$

$$P\left(\bigcup_{n=1}^{\infty} A_n \mid B\right) = \frac{P\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \cap B\right)}{P(B)}$$

pwise disjoint

$$= \frac{P\left(\bigcup_{n=1}^{\infty} (A_n \cap B)\right)}{P(B)}$$

$$= \frac{\sum_{n=1}^{\infty} P(A_n \cap B)}{P(B)}$$

$$= \sum_{n=1}^{\infty} \frac{P(A_n \cap B)}{P(B)}$$

$$= \sum_{n=1}^{\infty} P(A_n \mid B) \quad \checkmark$$

Exercise: Consider the probability space $(\Omega, \mathcal{B}, P)$ for rolling two dice:

$$\Omega = \left\{ \begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

C

$\mathcal{B} = \{$all subsets of Ω, including Ω itself$\}$

$(A \cup B) \cap C$

$P(A) = \#\{$points in $A\}/36$ for all $A \in \mathcal{B}$.

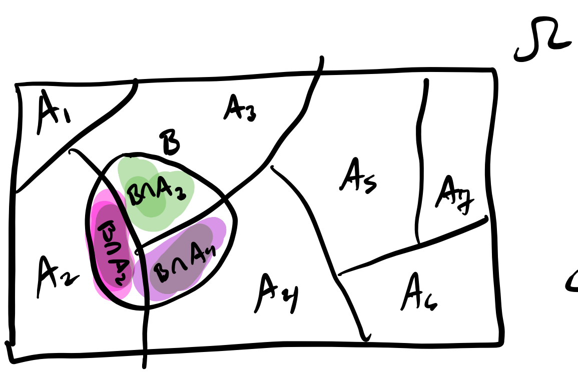
Let $A = \{$roll doubles$\}$, $B = \{$absolute difference in rolls less than 2$\}$, and $C = \{$sum of rolls 10 or more$\}$ and find

1. $P(B) = 16/36$
2. $P(A|B) = P(A \cap B)/P(B) = \dfrac{6/36}{16/36} = 6/16$
3. $P(A \cup B|C) = \dfrac{P((A \cup B) \cap C)}{P(C)} = \dfrac{4/36}{6/36} = \dfrac{4}{6} = \dfrac{2}{3}$

$P(B|A_n)P(A_n)$

$$P(B) = \sum_{n=1}^{\infty} P(B \cap A_n) = \sum_{n=1}^{\infty} P(B|A_n)\,P(A_n)$$

[Law of total probability]



$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

$$P(B|A_i) = \frac{P(A_i \cap B)}{P(A_i)}$$

$\Leftrightarrow$

$$P(A_i \cap B) = P(A_i|B)P(B)$$

$$P(A_i \cap B) = P(B|A_i)P(A_i)$$

Bayes' Rule

Let $A_1, A_2, \dots \in \mathcal{B}$ be a partition of the sample space and let $B \in \mathcal{B}$ be any set with $P(B) > 0$. Then

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{P(B)}$$

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)P(A_j)}.$$

This is called *Bayes' rule*.

Note $P(B) = \sum_{j=1}^{\infty} P(B|A_j)P(A_j) = \sum_{j=1}^{\infty} P(B \cap A_j)$ is the *law of total prob*.

Baby Bayes'

$\underbrace{A}_{A_1}, \underbrace{A^c}_{A_2}$

If A and B are events in $\mathcal{B}$ and $P(B) > 0$, then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$I = \{\text{infection}\}$

$T = \{\text{positive tests}\}$

Exercise: Consider an imperfect test for the presence of an infection such that

$$P(T|I) = 0.92 \quad \text{(Sensitivity)}$$
$$P(T^c|I^c) = 0.98 \quad \text{(Specificity)}$$

$P(I) = 0.05$

Moreover, suppose the infection is present in 5% of the population.

For an individual drawn at random from the population, find

1. $P(I|T)$ = P(Infected | Tested positive)
2. $P(I^c|T^c)$ = P(Not infected | Tested negative)

① $P(I|T) = \dfrac{\overbrace{P(T|I)}^{\text{Sens.}}\,P(I)}{P(T|I)\,P(I) + \underbrace{P(T|I^c)}_{1-\underbrace{P(T^c|I^c)}_{\text{spec}} = 1-0.98 = 0.02}\,\overbrace{P(I^c)}^{1-P(I) = 1-0.05 = .95}}$

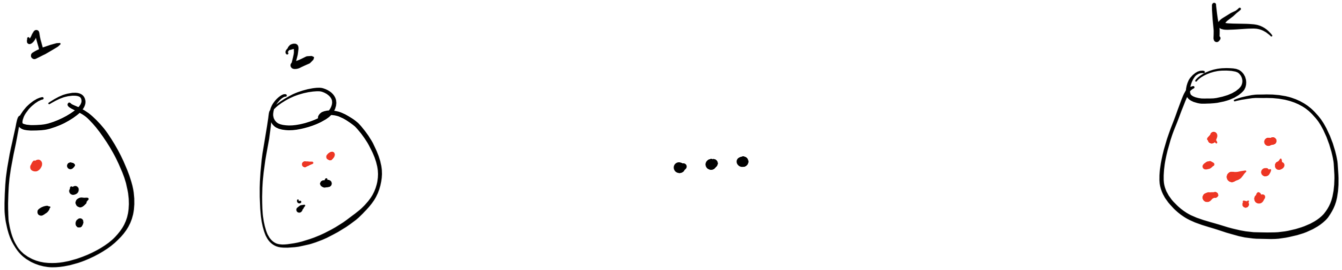
$= \dfrac{(0.92)\,0.05}{(0.92)\,0.05 + (0.02)\,0.95}$

$= 0.7077$

② $P(I^c|T^c) = \dfrac{\overbrace{P(T^c|I^c)}^{\text{spec}}\,\overbrace{P(I^c)}^{.95}}{\underbrace{P(T^c|I^c)P(I^c)} + \underbrace{P(T^c|I)}_{1-\underbrace{P(T|I)}_{\text{Sens}} = 1-0.92 = 0.08}\,\underbrace{P(I)}_{0.05}}$

$= \dfrac{(0.98)\,0.95}{(0.98)\,0.95 + (0.08)\,0.05}$

$= .996.$



Exercise: Suppose there are K bags, each containing K marbles, and that k of the marbles in bag k are red, for $k = 1, \ldots, K$. Select a bag at random and draw a marble from it:

$B_k = \{ \text{draw from bag } k \}$

$R = \{ \text{draw red marble} \}$

1. What is the probability of drawing a red marble?
2. Given you drew a red marble, what is the probability you drew from bag k?

(1) $$P(R) = \sum_{k=1}^{K} P(R \cap B_k) \leftarrow$$ [Law of total probability]

$B_1, \ldots, B_K$ are a partition

$$= \sum_{k=1}^{K} \underbrace{P(R \mid B_k)}_{= \frac{k}{K}} \underbrace{P(B_k)}_{= \frac{1}{K}}$$

$$= \sum_{k=1}^{K} \frac{k}{K} \, \frac{1}{K}$$

$$= \frac{1}{K^2} \boxed{\sum_{k=1}^{K} k}$$

$$= \frac{1}{K^2} \frac{K(K+1)}{2}$$

$$= \frac{(K+1)}{2K}$$

② $$P(B_k \mid R) = \frac{\overbrace{P(R \mid B_k)\, P(B_k)}^{P(B_k \cap R)}}{\underbrace{\sum_{k=1}^{K} P(R \mid B_k)\, P(B_k)}_{P(R)}}$$

$$= \frac{\frac{k}{K} \; \frac{1}{K}}{\frac{K+1}{2K}}$$

$$= \frac{2k}{K(K+1)} .$$

Independence

Two events A and B in $\mathcal{B}$ are called *independent* if

$$P(A \cap B) = P(A)P(B).$$

Also: If A, B independent, so are the pairs of events A, B^c and A^c, B and A^c, B^c.

Equivalent definitions of independence

① => ②

If $P(A \cap B) = P(A)P(B)$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

The following statements are equivalent:

1. $P(A \cap B) = P(A)P(B)$
2. $P(A) = P(A|B)$
3. $P(B) = P(B|A)$

Prob of A is same whether B occurs or not.

Exercise: Given a probability space $(\Omega, \mathcal{B}, P)$ and events $A, B \in \mathcal{B}$, show:

1. If $P(A \cap B) = 0$ and $P(A) = 1$, then $P(B) = 0$.
2. If $P(A) > 0$, $P(B) > 0$, and $P(A) < P(A|B)$, then $P(B) < P(B|A)$.
3. If A and B are independent, then A^c and B^c are independent.

(1) $P(A \cap B) = 0$, $P(A) = 1 \Rightarrow P(B) = 0$.

$$P(A \cup B) = \underbrace{P(A)}_{=1} + P(B) - \underbrace{P(A \cap B)}_{=0}$$

$$1 \geq P(A \cup B) = 1 + \underbrace{P(B)}_{\geq 0} . \Rightarrow P(B) = 0.$$

$|A| = \#$ elements in A.

$\emptyset \quad \Omega$

Exercise: Let $\Omega = \{1, 2, \dots, n\}$ and A and B be (independently) randomly selected from the collection of all subsets of Ω. Show that $P(A \subset B) = (3/4)^n$.

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$$P(A \subset B) = \sum_{i=0}^{n} \sum_{j=0}^{n} P\left(A \subset B \cap \boxed{|A| = i} \cap \boxed{|B| = j}\right)$$

keep only $i \leq j$

Law of total prob

$$= \sum_{j=0}^{n} \sum_{i=0}^{j} \underbrace{P\left(A \subset B \cap |A| = i \cap |B| = j\right)}$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{j} P\left(A \subset B \,\middle|\, |A| = i \cap |B| = j\right) \underbrace{P\left(|A| = i \cap |B| = j\right)}_{P(|A| = i)\, P(|B| = j)}$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{j} \underbrace{P\left(A \subset B \mid |A|=i \wedge |B|=j\right)}_{\frac{\#\{\text{ways to draw } i \text{ elements from the } j \text{ in } B\}}{\#\{\text{ways to draw } i \text{ elements from } \Omega\}} = \frac{\binom{j}{i}}{\binom{n}{i}}} \underbrace{P(|A|=i)}_{\frac{\binom{n}{i}}{2^n}} \underbrace{P(|B|=j)}_{\frac{\#\{\text{subsets of } \Omega \text{ of size } j\}}{\#\{\text{subsets of } \Omega\}} = \frac{\binom{n}{j}}{2^n}}$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{\binom{j}{i}}{\cancel{\binom{n}{i}}} \frac{\cancel{\binom{n}{i}}}{2^n} \frac{\binom{n}{j}}{2^n}$$

$$= \left(\frac{1}{4}\right)^n \sum_{j=0}^{n} \binom{n}{j} \underbrace{\sum_{i=0}^{j} \binom{j}{i} 1^i \, 1^{j-i}}_{(1+1)^j}$$

$$= \left(\frac{1}{4}\right)^n \underbrace{\sum_{j=0}^{n} \binom{n}{j} 2^j \, 1^{n-j}}_{(2+1)^n}$$

$$= \left(\frac{3}{4}\right)^n .$$

Binomial Thm:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Multiplication rule

For any $A_1, A_2, \dots, A_n \in \mathcal{B}$, we have

$$P\left(\cap_{i=1}^n A_i\right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_n| \cap_{i=1}^{n-1} A_i).$$

Exercise: Prove the result by induction.

Exercise: Find prob. of hand $\boxed{10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit}$ from a 52-card deck.

$P(10) = 1/52$

$P(J|10) = 1/51$

$P(Q|10 \cap J) = 1/50$

$P(K|10 \cap J \cap Q) = 1/49$

$P(A|10 \cap J \cap Q \cap K) = 1/48$

$P(10 \cap J \cap Q \cap K \cap A) = \frac{1}{52 \cdot 51 \cdot \ldots \cdot 48} = \frac{(52-5)!}{52!}$

$P(\text{hand}) = \frac{5! \cdot (52-5)!}{52!} = \frac{1}{\binom{52}{5}}.$

For all $n \geq 2$

$$P\left(\cap_{i=1}^{n} A_i\right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_n| \cap_{i=1}^{n-1} A_i).$$

Prove by induction:

case $n=2$: $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 | A_1)$.

$\boxed{\checkmark}$ True.

show that case $n \Rightarrow$ case $n+1$.

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) = P\left(\left(\bigcap_{i=1}^{n} A_i\right) \cap A_{n+1}\right)$$

$$= P\left(\bigcap_{i=1}^{n} A_i\right) \cdot P\left(A_{n+1} \,\middle|\, \bigcap_{i=1}^{n} A_i\right)$$

$$= \underbrace{P(A_1) \cdot P(A_2 | A_1) \cdot \ldots \cdot P\left(A_n \,\middle|\, \bigcap_{i=1}^{n-1} A_i\right)}_{} P\left(A_{n+1} \,\middle|\, \bigcap_{i=1}^{n} A_i\right)$$

↑ This is the $n+1$ case, so we are done.

Mutual independence

A, B indep if $P(A \cap B) = P(A)P(B)$

A collection of events $A_1, A_2, \ldots, A_n$ are

1. *mutually independent* if for any subcollection $A_{i_1}, \ldots, A_{i_K}$, we have

$$P\left(\bigcap_{j=1}^{K} A_{i_j}\right) = \prod_{j=1}^{K} P(A_{i_j})$$

$\cancel{\Uparrow}$ $\Downarrow$ ✓

2. *pairwise independent* if $P(A_j \cap A_i) = P(A_j)P(A_i)$ for all $i \neq j$.

Extend independence between two events to independence among ≥ 2 events.

Note: Can have pairwise indep. without mutual independence (ex 1.3.11 of CB).

Exercise: Find the probability of rolling 3 sixes in 8 rolls of a die.

$A_1, A_2, \ldots A_8$ $\quad$ A_i : roll ⚅ on roll i $\quad$ $P(A_i) = \frac{1}{6}$

Assume $A_1, \ldots A_8$ mutually indep.

$$P(A_1 \cap A_2^c \cap A_3^c \cap A_4 \cap A_5^c \cap A_6^c \cap A_7^c \cap A_8)$$

$$= P(A_1)\, P(A_2^c)\, P(A_3^c) \cdot P(A_4) \cdot P(A_5^c) \cdot P(A_6^c) \cdot P(A_7^c) \cdot P(A_8)$$

$$= \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^5$$

$$P(\text{roll } 3 \text{ ⚅'s in } 8 \text{ rolls}) = \binom{8}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^5$$

By counting:

$$P(\text{roll } 3 \text{ ⚅'s in } 8 \text{ rolls}) = \frac{\#\{\text{ways to roll } 3 \text{ ⚅'s}\}}{\#\{\text{ways to roll } 8 \text{ times}\}}$$

$$= \frac{\binom{8}{3}\, 5^5}{6^8}$$

$$= \binom{8}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^5 .$$