STAT 712 fa 2022 Lec 5 slides

Expected value, variance, moment generating functions

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Expected value of a random variable

The *expected value* $\mathbb{E}X$ of a random variable X is defined as

$$\mathbb{E}X = \begin{cases} \displaystyle\sum_{x \in \mathcal{X}} x \cdot p_X(x) & \text{if } X \text{ discrete with pmf } p_X \text{ and support } \mathcal{X} \\ \displaystyle\int_{-\infty}^{\infty} x \cdot f_X(x) dx & \text{if } X \text{ continuous with pdf } f_X \end{cases}$$

- The average of many realizations of X should be close to $\mathbb{E}X$.
- $\mathbb{E}X$ is the "balancing point" of the pmf/pdf.
- We often use μ_X to denote $\mathbb{E}X$.
- We often call $\mathbb{E}X$ the *mean* of X

Exercise: Let $X =$ up-face of one roll of a K-sided die. Find $\mathbb{E}X$.

Exercise: Let $X \sim f_X(x) = \mathbf{1}(0 \leq x \leq 1)$. Find $\mathbb{E}X$.

Expected value of a function of a random variable

The expected value $\mathbb{E}g(X)$ of the rv $g(X)$, where g is any function, is

$$\mathbb{E}g(X) = \begin{cases} \sum_{x \in \mathcal{X}} g(x) \cdot p_X(x) & \text{if } X \text{ discrete with pmf } p_X \text{ and support } \mathcal{X} \\ \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx & \text{if } X \text{ continuous with pdf } f_X \end{cases}$$

Theorem (Expected value results)

Let X be a rv such that $\mathbb{E}g_1(X)$ and $\mathbb{E}g_2(X)$ exist. For $a, b, c \in \mathbb{R}$ we have:

1. $\mathbb{E}(ag_1(X) + bg_2(X) + c) = a\mathbb{E}g_1(X) + b\mathbb{E}g_2(X) + c$
2. *If $g_1(x) \leq g_2(x)$ for all $x \in \mathbb{R}$ then $\mathbb{E}g_1(X) \leq \mathbb{E}g_2(X)$.*
3. *If $a \leq g_1(x) \leq b$ for all $x \in \mathbb{R}$ then $a \leq \mathbb{E}g_1(X) \leq b$.*

Prove the results.

Exercise: Let $X \sim f_X(x) = 2x^{-3}\mathbf{1}(x \geq 1)$. Find $\mathbb{E}\sqrt{X}$.

Variance of a random variable

The *variance* $\mathrm{Var}\, X$ of a random variable X is defined as

$$\mathrm{Var}\, X = \mathbb{E}(X - \mu_X)^2,$$

where $\mu_X = \mathbb{E}X$.

- $\mathrm{Var}\, X$ is the expected squared deviation of X from μ_X.
- Measure of "spread" for the distribution of X.
- Often use σ_X^2 to denote $\mathrm{Var}\, X$.
- Use σ_X to denote $\sqrt{\mathrm{Var}\, X}$, which is called the *standard deviation* of X.

Useful expression: $\mathrm{Var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

Exercise: Let $X \sim p_X(x) = p^x(1-p)^{1-x} \cdot \mathbf{1}(x \in \{0,1\})$, $p \in (0,1)$. Find $\operatorname{Var} X$.

Theorem (Mean and variance of shifted and scaled random variables)

Let X be an rv with finite mean and variance. Then for any constants a and b

- $\mathbb{E}(aX + b) = a\mathbb{E}X + b$
- $\operatorname{Var}(aX + b) = a^2 \operatorname{Var} X$

Prove the result.

Exercise: Let $X \sim f_X(x) = \mathbf{1}(0 \leq x \leq 1)$ and suppose $Y = 4X - 2$.

1. Give $\mathbb{E}Y$.
2. Give $\operatorname{Var} Y$.

Theorem (Марков's inequality)

For any nonnegative rv X we have

$$P_X(X \geq a) \leq \frac{\mathbb{E}X}{a}$$

for all $a > 0$.

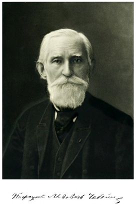
Prove the result.

Theorem (Чебышёв's inequality)

For any rv X with mean μ_X and var. $\sigma_X^2 < \infty$ and any constant $K > 0$, we have

$$P_X(|X - \mu_X| < K\sigma_X) \geq 1 - \frac{1}{K^2}.$$

- Any rv X lies within K st'd dev's of its mean with prob. at least $1 - 1/K^2$.
- E.g. any rv X lies within 4 st'd dev's of its mean at least 93.75% of the time.



Prove the result.

Moments about the origin and about the mean

Let X be a random variable. For each integer k, the

- *kth moment about the origin* of X is $m_k = \mathbb{E}X^k$
- *kth moment about the mean* of X is $m'_k = \mathbb{E}(X - m_1)^k$

Usually refer to moments about the origin simply as *moments*.

Also refer to moments about the mean as *central moments*.

$\text{Var}\, X = m'_2 = m_2 - (m_1)^2$ = the 2nd moment minus the 1st moment squared

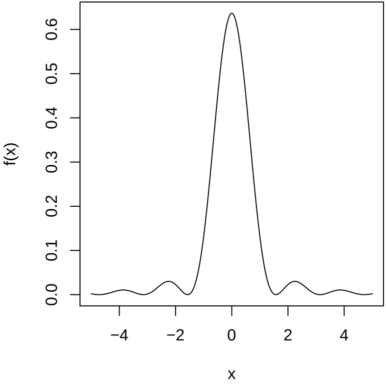
Sometimes moments do not exist (diverge to $\pm\infty$ or are like "$\infty \pm \infty$"):

Example: Let $X \sim f_X$, where

$$f_X(x) = \begin{cases} \dfrac{\lambda}{\pi}\dfrac{\sin^2(x/\lambda)}{x^2}, & x \neq 0 \\ \dfrac{1}{\pi\lambda}, & x = 0. \end{cases}$$

for some $\lambda > 0$.

Moments $\mathbb{E}X$ and $\mathbb{E}X^2$ e.g. do not exist.



Moment-generating function

The *moment-generating function (mgf)* M_X of a rv X is the function given by

$$M_X(t) = \mathbb{E}e^{tX},$$

provided the expectation is finite for t in a neighborhood of 0.

"A moment-generating function is a function that generates moments."

–Dr. Josh Tebbs

Theorem (generating moments with the moment-generating function)

If X is a rv with mgf M_X, then

$$\mathbb{E}X^k = M_X^{(k)}(0),$$

where

$$M_X^{(k)}(0) = \left(\frac{d}{dt}\right)^k M_X(t)\Big|_{t=0}.$$

Recipe: To get the kth moment we

1. differentiate the mgf k times with respect to t,
2. evaluate the result at $t = 0$.

Prove the result.

Exercise: Let $X \sim \text{Exponential}(\lambda)$, so that

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \mathbf{1}(x > 0)$$

for some $\lambda > 0$.

1. Find the mgf of X.
2. Use the mgf of X to find $\mathbb{E}X$ and $\operatorname{Var} X$.

Exercise: Let $X \sim \text{Binomial}(n, p)$, so that

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, \dots, n$$

for some $p \in (0, 1)$.

1. Find the mgf of X.
2. Use the mgf of X to find $\mathbb{E}X$ and $\text{Var}\, X$.

Mgfs really more useful for characterizing distributions than for getting moments.

Theorem (identification of distribution by mgf)

Suppose $X \sim F_X$, $Y \sim F_Y$ *and* $\mathbb{E}X^k < \infty$, $\mathbb{E}Y^k < \infty$ *for all* $k = 1, 2, \ldots$ *Then:*

- M_X *and* M_Y *exist and* $M_X(t) = M_Y(t)$ $\forall t$ *in a n'hood of* $0 \iff X \overset{d}{=} Y$.
- *If* X, Y *have bounded support,* $\mathbb{E}X^k = \mathbb{E}Y^k$ $\forall k = 1, 2, \ldots \iff X \overset{d}{=} Y$.

So rvs with the same mgf have the same distribution; the mgf identifies the dist.

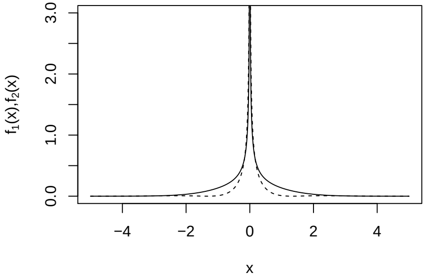
Note:

1. M_X exists $\overset{\not\Leftarrow}{\Longrightarrow}$ $\mathbb{E}X^k < \infty$ for all $k = 1, 2, \ldots$
2. In general, $\mathbb{E}X^k = \mathbb{E}Y^k$, $k = 1, 2, \ldots \not\Longrightarrow X \overset{d}{=} Y$

Example: The rvs $X \sim f_X$ and $Y \sim f_Y$, with

$$f_X(x) = \frac{1}{3\sqrt{\pi}}|x|^{-2/3}\exp(-|x|^{2/3})$$
$$f_Y(y) = f_X(y)(1 + 0.5 \cdot (\cos(\sqrt{3}|x|^{2/3}) - \sqrt{3}\sin(\sqrt{3}|x|^{2/3})))$$

are such that $\mathbb{E}X^k = \mathbb{E}Y^k$ for each $k = 1, 2, \ldots$ See Berg (1988)



Existence of abs. moment implies existence of lower order moments

If $\mathbb{E}|X|^m < \infty$ then $\mathbb{E}X^k$ exists and is finite for all $k \leq m$.

Prove the result.

Theorem

Let X have mgf $M_X(t)$. Then for any constants a and b the mgf of $Y = aX + b$ is

$$M_Y(t) = e^{bt} M_X(at).$$

Prove the result.

Exercise: Let $X \sim \text{Poisson}(\lambda)$, i.e. $X \sim p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \cdot \mathbf{1}(x \in \{0, 1, 2, \dots\})$.

1. Find the mgf of X.
2. Find the mgf of $Y = (X - \lambda)\lambda^{-1/2}$.

Convergence of a sequence of mgfs

Let $X_1, X_2, \ldots$ be rvs with cdfs $F_1, F_2, \ldots$ and mgfs $M_{X_1}, M_{X_2}, \ldots$ such that

$$\lim_{n \to \infty} M_{X_n}(t) = M_X(t) \quad \text{for all } t \text{ in a neighborhood of } 0,$$

where M_X is an mgf with corresponding cdf F_X. Then

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

for all x at which F_X is continuous.

We will later use mgfs to prove a version of the Central Limit Theorem! 

Exercise: For each $n \geq 1$ let $X_n \sim \text{Poisson}(\lambda_n)$ and $Y_n = (X_n - \lambda_n)\lambda_n^{-1/2}$. Show that if $\lambda_n \to \infty$ as $n \to \infty$ then

$$\lim_{n \to \infty} M_{Y_n}(t) = e^{t/2} \;\; (= \text{mgf of Normal}(0,1) \text{ distribution})$$

Christian Berg. The cube of a normal distribution is indeterminate. *The Annals of Probability*, pages 910–913, 1988.