STAT 712 fa 2022 Lec 9 slides

Conditional distributions, independence

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

1 Conditional pmfs and pdfs

2 Conditional expectation and variance

3 Independence of random variables

$$P(X=x \mid Y=y) = \frac{P(\{X=x\} \cap \{Y=y\})}{P(Y=y)} = \frac{p(x,y)}{p_Y(y)}$$

Conditional probability mass functions

Let (X, Y) be discrete rvs with joint pmf p and marginal pmfs p_X and p_Y.

- For any y such that $p_Y(y) > 0$, the *conditional pmf* of $X|Y = y$ is

$$p(x|y) = \frac{p(x, y)}{p_Y(y)}, \quad x \in \mathbb{R}.$$

$$p(y|x) = \frac{p(x,y)}{p_X(x)}$$

- Likewise the conditional pmf of $Y|X = x$.

Exercise: Show that $p(x|y)$ is a valid pmf.

(*)

(i) $p(x|y) \geq 0$

(ii) $\sum_{x \in \mathcal{X}} p(x|y) = 1$

$$\sum_{x \in \mathcal{X}} p(x|y) = \sum_{x \in \mathcal{X}} \frac{p(x,y)}{p_Y(y)} = \frac{p_Y(y)}{p_Y(y)} = 1$$

Let X = sum of two dice rolls, Y = max.

$\mathcal{Y}$

$\mathcal{X}$		1	2	3	4	5	6	$p_X(x)$
	2	1/36						1/36
	3		2/36					2/36
	4		1/36	2/36				3/36
	5			2/36	2/36			4/36
	6			1/36	2/36	2/36		5/36
	7				2/36	2/36	2/36	6/36
	8				1/36	2/36	2/36	5/36
	9					2/36	2/36	4/36
	10					1/36	2/36	3/36
	11						2/36	2/36
	12						1/36	1/36
	$p_Y(y)$	1/36	3/36	5/36	7/36	9/36	11/36	

$p(x,y)$

Exercise: Tabulate $p(x|y=4)$ and $p(y|x=7)$.

y	4	5	6
$p(y\|x=7)$	1/3	1/3	1/3

$\circledast$

x	5	6	7	8
$p(x \mid y=4)$	$\frac{p(x=5, y=4)}{p(y=4)} = \frac{2/36}{7/36}$ $\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$

Conditional probability density functions

Let (X, Y) be continuous rvs with joint pdf f and marginal pdfs f_X and f_Y.

- For any y such that $f_Y(y) > 0$, the *conditional pdf* of $X|Y = y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad x \in \mathbb{R}.$$

- Likewise the conditional pdf of $Y|X = x$.

Exercise: Show that $f(x|y)$ is a valid pdf.

*

(i) $f(x|y) \geq 0$

(ii) $\int_{\mathbb{R}} f(x|y)\,dx = 1$ ←

$$\int_{\mathbb{R}} f(x|y)\,dx = \int_{\mathbb{R}} \frac{f(x,y)}{f_Y(y)}\,dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

Exercise: Let $(X, Y) \sim f(x, y) = y(1-x)^{y-1}e^{-y}\mathbf{1}(0 < x < 1, 0 < y < \infty)$.

1. Find the pdf $f(x|y)$ of X given $Y = y$.
2. Find $P(X < 1/2 | Y = 2)$.

(1)

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

(1)

Need $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx =$

$$= \int_0^1 y(1-x)^{y-1} e^{-y}\,dx$$

$\left(\Gamma(\alpha+1)=\alpha\Gamma(\alpha)\right) = y e^{-y} \underbrace{\int_0^1 x^{1-1}(1-x)^{y-1}\,dx}_{\text{Beta}(1,y) = \frac{\Gamma(1)\Gamma(y)}{\Gamma(1+y)}}$

$= \frac{\Gamma(1)\Gamma(y)}{\Gamma(1+y)} y e^{-y}$

$= \frac{1 \cdot \Gamma(y)}{y\Gamma(y)} y e^{-y}$

$= e^{-y}.$

For $x \in (0,1)$:

$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{y(1-x)^{y-1} e^{-y}}{e^{-y}} = \underline{y(1-x)^{y-1}}$

$f(x|y) = \frac{\Gamma(y+1)}{\Gamma(y)\Gamma(1)} x^{1-1}(1-x)^{y-1} \mathbb{1}(0<x<1)$ $=y$

$X \mid Y=y \sim \text{Beta}(\alpha=1, \beta=y)$

② $P(\underline{X<\tfrac{1}{2}} \mid Y=2) = \int_0^{1/2} f(x|y=2)\,dx$

$= \int_0^{1/2} 2(1-x)^{2-1}\,dx$

$$= 2\int_0^{1/2} (1-x)\,dx$$

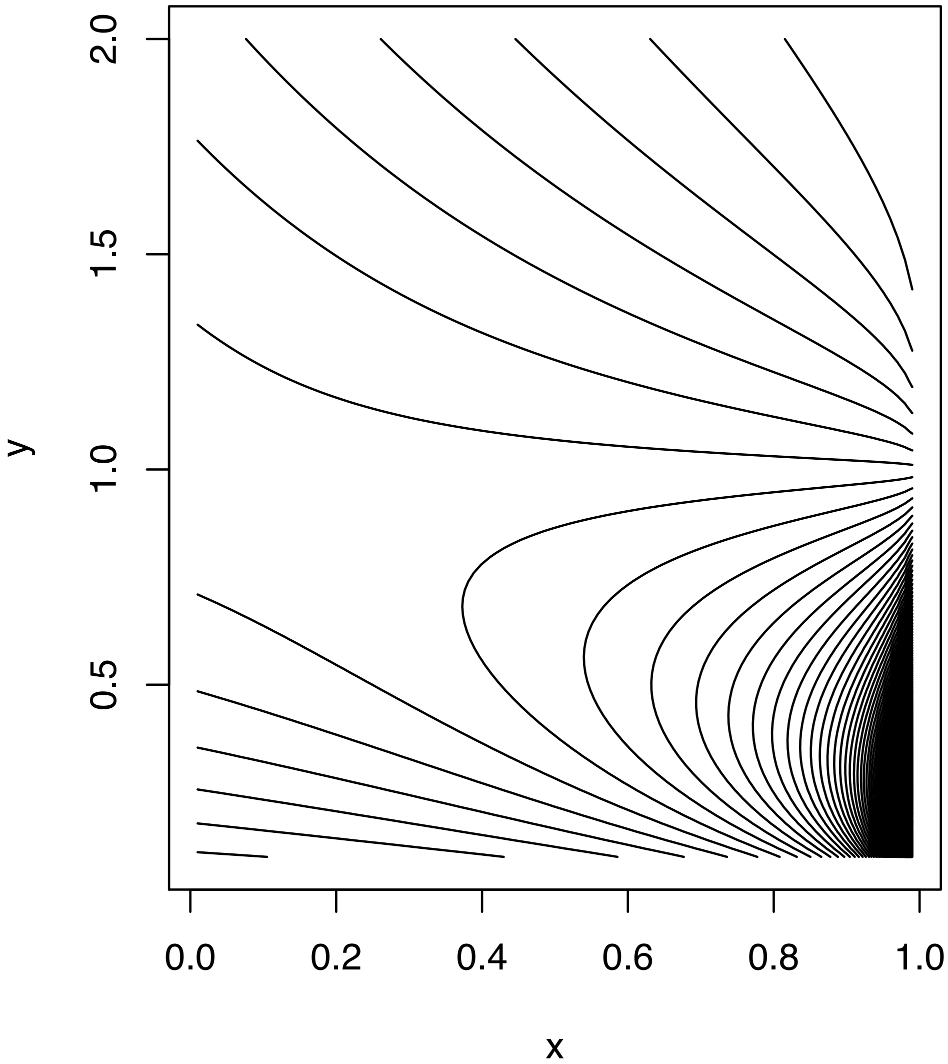
$$= 2\left(x - \frac{x^2}{2}\right)\Bigg|_0^{1/2}$$

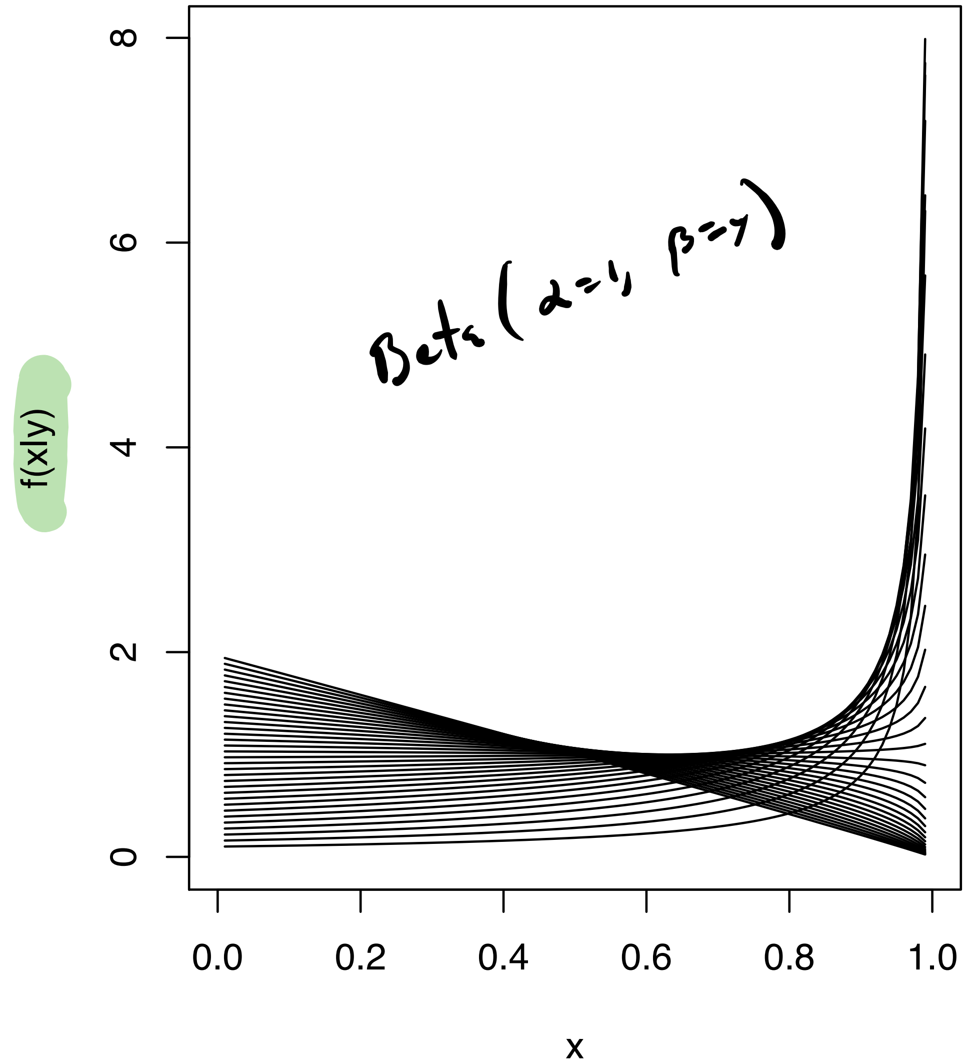
$$= 2\left(\frac{1}{2} - \frac{1}{8}\right)$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

contours of $f(x,y)$

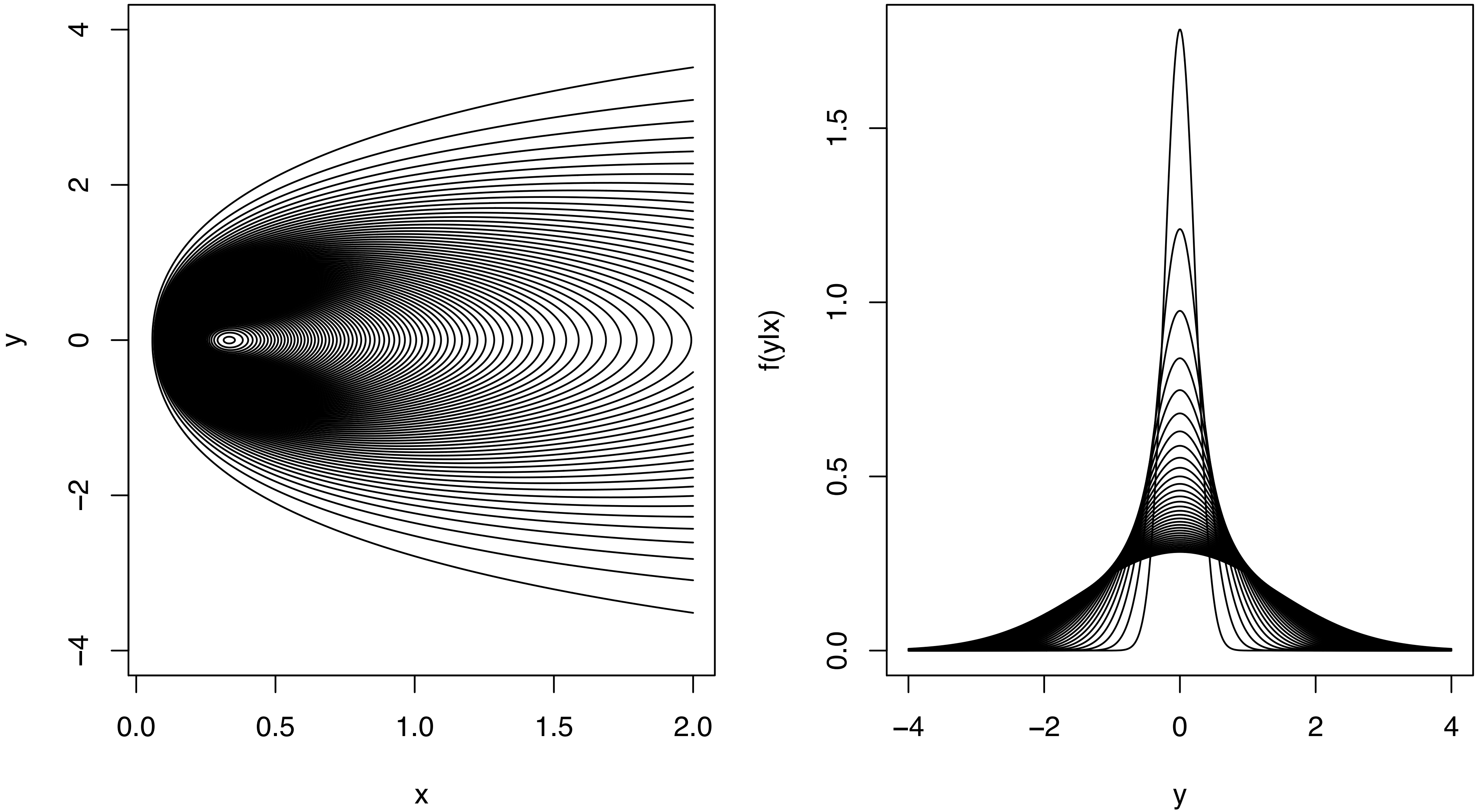




Exercise: Let (X, Y) have joint pdf given by

$$f(x, y) = \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2 + 1)\right] \mathbf{1}(0 < x < \infty, -\infty < y < \infty).$$

1. Find the conditional pdf $f(y|x)$ of Y given $X = x$.
2. Find $P(Y > 1|X = 4)$.



4
2
0
−2
−4
y
0.0
0.5
1.0
1.5
2.0
x
0.0
0.5
1.0
1.5
f(y|x)
−4
−2
0
2
4
y

Exercise: Let (U, V) be a pair of rvs with joint pdf given by

$$f(u, v) = 6(v - u)\mathbf{1}(0 < u < v < 1).$$

1. Find the conditional pdf $f(u|v)$ of U given $V = v$.
2. Find $P(U < 1/4|V = 1/2)$.

1. Conditional pmfs and pdfs
2. Conditional expectation and variance
3. Independence of random variables

Conditional expectation

Let (X, Y) be discrete or continuous rvs on $\mathcal{X}$ and $\mathcal{Y}$ and $g : \mathbb{R} \to \mathbb{R}$.
For any $y \in \mathcal{Y}$, the *conditional expectation* of $g(X)$ given $Y = y$ is

$$\mathbb{E}[g(X)|Y = y] = \begin{cases} \displaystyle\sum_{x \in \mathcal{X}} g(x) \cdot p(x|y) & \text{for } (X, Y) \text{ discrete} \\ \displaystyle\int_{\mathbb{R}} g(x) \cdot f(x|y) & \text{for } (X, Y) \text{ continuous} \end{cases}$$

The conditional expectation $\mathbb{E}[g(Y)|X = x]$ is likewise defined.

Note that $\mathbb{E}[g(X)|Y = y]$ is a function of y, since X is summed/integrated out.

We often have $g(x) = x$, so that we consider $\mathbb{E}[X|Y = y]$.

← a function of Y.

If we write $\mathbb{E}[X|Y]$, without specifying a value y for Y, then $\mathbb{E}[X|Y]$ is a rv.

Exercise: Let $(X, Y) \sim f(x, y) = y(1 - x)^{y-1} e^{-y} \mathbf{1}(0 < x < 1, 0 < y < \infty)$.

1. Find $\mathbb{E}[X|Y]$.
2. Find $\mathbb{E}[X|Y = 2]$.

$X | Y=y \sim \text{Beta}(1, y)$

①

$$\mathbb{E}[X|Y=y] = \int_0^1 x \cdot y(1-x)^{y-1}\, dx$$

$\vdots$

$$= \frac{1}{1+y}$$

② $\mathbb{E}[X|Y=2] = \frac{1}{1+2} = \frac{1}{3}$

$$\mathbb{E}[X|Y] = \frac{1}{1+Y} \longleftarrow \text{random variable.}$$

Exercise: Let (U, V) be a pair of rvs with joint pdf given by

$$f(u, v) = 6(v - u)\mathbf{1}(0 < u < v < 1).$$

1. Find $\mathbb{E}[U|V]$.
2. Find $\mathbb{E}[U|V = 1/2]$.

$$\mathrm{Var}\,X = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

Conditional variance

The conditional variance of X given that $Y = y$ is

$$\mathrm{Var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y].$$

If we write $\mathrm{Var}[X|Y]$, without specifying a value y for Y, then $\mathrm{Var}[X|Y]$ is a rv.

Useful expression: $\mathrm{Var}[X|Y] = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Exercise: Let $(X, Y) \sim f(x, y) = y(1-x)^{y-1}e^{-y}\mathbf{1}(0 < x < 1, 0 < y < \infty)$.

1. Find $\text{Var}[X|Y]$.
2. Find $\text{Var}[X|Y = 2]$.

$X | Y \sim \text{Beta}(1, Y)$

$$\text{Var}[X|Y] = \frac{Y}{(1+Y)^2(1+Y+1)}$$

$$\text{Var}[X|Y=2] = \frac{2}{(1+2)^2(1+2+1)} = \frac{2}{9 \cdot 4} = \frac{1}{18}.$$

Exercise: Let (X, Y) have joint pdf given by

$$f(x,y) = \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2+1)\right] \mathbf{1}(0 < x < \infty, -\infty < y < \infty).$$

1. Find $\mathbb{E}[Y|X]$.
2. Find $\mathrm{Var}[Y|X]$.
3. Find $\mathrm{Var}[Y|X=2]$.

1 Conditional pmfs and pdfs

2 Conditional expectation and variance

3 Independence of random variables

Events A, B indep means $P(A \cap B) = P(A)P(B)$

equiv: $P(A | B) = P(A)$

$P(B | A) = P(B)$

On $(\Omega, \mathcal{B}, P)$, independence of events $A, B \in \mathcal{B}$ is the property that

$$P(A \cap B) = P(A)P(B).$$

Independence of random variables

Two rvs X and Y are *independent* if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B) \quad \text{for all } A, B \in \mathcal{B}(\mathbb{R}).$$

We sometimes write $X \perp\!\!\!\perp Y$ to express that X and Y are independent.

Basically: If all pairs of events concerning X and Y are independent, then $X \perp\!\!\!\perp Y$.

$p(x,y) = P[X=x \cap Y=y] = P(X=x)\,P(Y=y)$

so events $\{X=x\}$, $\{Y=y\}$ are indep for all x,y.

Theorem (Independence iff joint is product of marginals)

- *If* (X,Y) *discrete with joint pmf* p *and marginal pmfs* p_X *and* p_Y, *then*

$$X \perp\!\!\!\perp Y \iff \boxed{p(x,y) = \boxed{p_X(x)p_Y(y)}} \text{ for all } (x,y) \in \mathbb{R}^2.$$

- *If* (X,Y) *continuous with joint pdf* f *and marginal pdfs* f_X *and* f_Y, *then*

$$X \perp\!\!\!\perp Y \iff f(x,y) = f_X(x)f_Y(y) \text{ for all } (x,y) \in \mathbb{R}^2.$$

Check independence by checking whether the joint is the product of the marginals.

Exercise: Suppose there are six chairs in a circle numbered $1, \ldots, 6$. Then:

1. Let X be the roll of a die and sit in chair X.
2. Roll die again and move that many chairs clockwise.
3. Let Y be the number of the chair in which you now sit.

Are X and Y independent?

Exercise: Let X and Y be independent rvs with marginal pmfs given by

$$p_X(x) = p^x(1-p)^{1-x} \cdot \mathbf{1}(x \in \{0,1\})$$
$$p_Y(y) = \binom{3}{y}\eta^y(1-\eta)^{3-y} \cdot \mathbf{1}(y \in \{0,1,2,3\})$$

Give the joint pmf of the rv pair (X, Y).

④ $p(x,y) = p^x(1-p)^{1-x}\binom{3}{y}\eta^y(1-\eta)^{3-y}\,\mathbb{1}(x\in\{0,1\})\,\mathbb{1}(y\in\{0,1,2,3\})$

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = \frac{6}{5}[1 - (x - y)^2] \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

$*$

$$f_X(x) = \int_0^1 \frac{6}{5}\left(1 - (x^2 - 2xy + y^2)\right) dy$$

$$= \frac{6}{5} \int_0^1 \left(1 - x^2 + 2xy - y^2\right) dy$$

$$= \frac{6}{5} \left[y - yx^2 + xy^2 - \frac{y^3}{3}\right]\Big|_0^1$$

$$= \frac{6}{5}\left[1 - x^2 + x - \frac{1}{3}\right] \quad \text{for} \quad x \in (0,1).$$

$$f_Y(y) = \frac{6}{5}\left[1 - y^2 + y - \frac{1}{3}\right] \quad \text{for} \quad y \in (0,1)$$

We have

$$f(x,y) \neq f_X(x) f_Y(y),$$

so $\quad X \not\perp Y.$

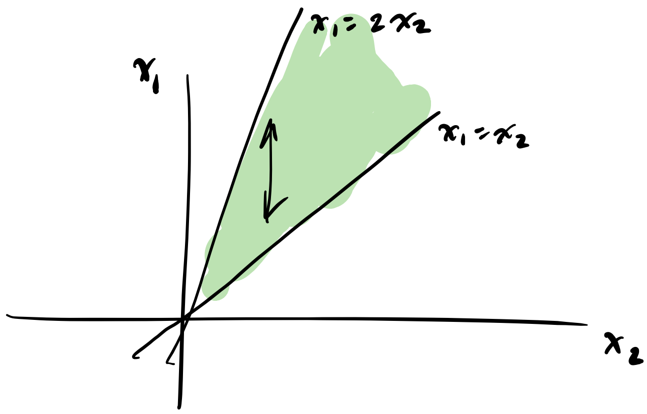
$$f_{X_1}(x_1) = e^{-x_1}\,\mathbb{1}(x_1 > 0)$$
$$f_{X_2}(x_2) = e^{-x_2}\,\mathbb{1}(x_2 > 0)$$

Exercise: Let X_1 and X_2 be independent rvs with the Exponential(1) distribution.

1. Give the joint pdf of (X_1, X_2)
2. Find $P(X_2 < X_1 < 2X_2)$. ...

* ① $f(x_1, x_2) = e^{-x_1 - x_2} \underbrace{\mathbb{1}(x_1 > 0) \cdot \mathbb{1}(x_2 > 0)}_{\mathbb{1}(x_1 > 0,\ x_2 > 0)}$

② $P(X_2 < X_1 < 2X_2) = \int_0^\infty \int_{x_2}^{2x_2} e^{-x_1 - x_2}\, dx_1\, dx_2 = \cdots$



x_1
$x_1 = 2x_2$
$x_1 = x_2$
x_2

$$p(x,y) = p_X(x)\, p_Y(y)$$
$$f(x,y) = f_X(x)\, f_Y(y)$$

Theorem (Easier independence check)

Let (X, Y) be discrete or continuous rvs with joint pmf p or joint pdf f.

Then $X \perp\!\!\!\perp Y \iff$ there exist functions g and h such that

$$p(x,y) = g(x)h(y) \quad \text{for all } (x,y) \in \mathbb{R}^2 \quad \text{or}$$
$$f(x,y) = g(x)h(y) \quad \text{for all } (x,y) \in \mathbb{R}^2, \quad \text{respectively.}$$

Check if the joint is the product of a function of just x and a function of just y.

Exercise: Let (X, Y) have the joint pdf

$$f(x, y) = 18xy(y - xy) \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

$$f(x,y) = \underbrace{18 \cdot x(1-x) \cdot \mathbb{1}(0<x<1)}_{g(x)} \cdot \underbrace{y^2 \cdot \mathbb{1}(0<y<1)}_{h(y)}$$

So $X \perp\!\!\!\perp Y$.

$\frac{6}{5}\left[1 - x^2 + 2xy - y^2\right]$

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = \frac{6}{5}[1 - (x - y)^2] \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

* Can I find $g(x)$, $h(y)$ s.t.

$f(x,y) = g(x) \cdot h(y)$?

No, $\Rightarrow$ $X \not\perp Y$.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = \frac{1}{8}(x + y) \cdot \mathbf{1}(0 < x < 2, 0 < y < 2).$$

Check whether X and Y are independent.

cannot factorize NO.

Exercise: Let (X, Y) have the joint pdf

$$f(x,y) = \frac{1}{4\pi}\exp\left[-\frac{x^2+y^2-2x+1}{4}\right] \quad \text{for all } x, y \in \mathbb{R}.$$

Check whether X and Y are independent.

$$f(x,y) = \underbrace{\frac{1}{4\pi} e^{-\frac{y^2}{4}}}_{h(y)} \cdot \underbrace{e^{-\frac{x^2-2x+1}{4}}}_{g(x)}$$

yes, $X \perp\!\!\!\perp Y$.

$P(A \cap B) = P(A)P(B)$

A, B ind $\Leftrightarrow$ $P(A|B) = P(A)$

$P(B|A) = P(B)$

$X \perp\!\!\!\perp Y \Leftrightarrow$ $f(x,y) = f_X(x) f_Y(y)$

$f(x|y) = f_X(x)$

$f(y|x) = f_Y(y)$

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = y(1-x)^{y-1} e^{-y} \mathbf{1}(0 < x < 1, 0 < y < \infty)$$

Check whether X and Y are independent.

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No factorization. $X \not\perp\!\!\!\perp Y$.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2 + 1)\right] \mathbf{1}(0 < x < \infty, -\infty < y < \infty)$$

Check whether X and Y are independent.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = 4xy \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

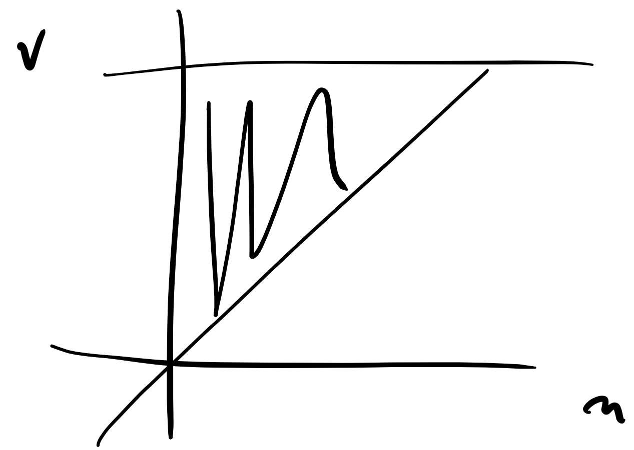
Check whether X and Y are independent.

Exercise: Let (U, V) be a pair of rvs with joint pdf given by

$$f(u, v) = 6(v - u)\boxed{\mathbb{1}(0 < u < v < 1).}$$

Check whether U and V are independent.

NO



Theorem (Expectation of the product of independent rvs)

Let X and Y be independent rvs. Then

$$\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y.$$

Expectation of a product is a product of expectations.

Moreover, for any functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}g(X)h(Y) = \mathbb{E}g(X)\mathbb{E}h(Y).$$

Exercise: Prove the result.

Let $X \perp\!\!\!\perp Y$ $\quad (X, Y) \sim f(x,y)$ with $X \sim f_X$, $Y \sim f_Y$.

Then

$$\mathbb{E}\, g(X)\, h(Y) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x)\, h(y)\, \underbrace{f(x,y)}_{f_X(x)\, f_Y(y)}\, dx\, dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} g(x)\, h(y)\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \underbrace{\int_{\mathbb{R}} g(x)\, f_X(x)\, dx}\ \underbrace{\int_{\mathbb{R}} h(y)\, f_Y(y)\, dy}$$

$$= \mathbb{E}\, g(X)\ \mathbb{E}\, h(Y)$$

Exercise: Let X and Y be independent rvs with marginal pdfs

$$f_X(x) = 2e^{-2x}\mathbf{1}(x > 0)$$
$$f_Y(y) = e^{-2|y|}\mathbf{1}(-\infty < y < \infty)$$

Find $\mathbb{E}XY^2$.

[just get started]

(i) $\mathbb{E}XY^2 = \int_{-\infty}^{\infty}\int_{0}^{\infty} xy^2 \underbrace{2e^{-2x} e^{-2|y|}}_{f(x,y)} \, dx \, dy$

OR

(ii) $\mathbb{E}XY^2 = \mathbb{E}X \, \mathbb{E}Y^2 = \int_0^\infty x \, 2e^{-2x} \, dx \cdot \int_{-\infty}^{\infty} y^2 e^{-2|y|} \, dy$

Beta(α, β) has mean $\frac{\alpha}{\alpha+\beta}$

Exercise: Let X and Y be independent rvs such that

$$X \sim \text{Beta}(3,4)$$
$$Y \sim \text{Beta}(1,2)$$

Find $\mathbb{E}[XY]$.

$\mathbb{E}XY = \mathbb{E}X \cdot \mathbb{E}Y = \frac{3}{3+4} \cdot \frac{1}{1+2}$.

Theorem (mgf of a sum of independent random variables)

If X and Y are indep. rvs with mgfs M_X and M_Y, the mgf of $V = X + Y$ is

$$M_V(t) = M_X(t)M_Y(t)$$

for all t at which $M_X(t)$ and $M_Y(t)$ are defined.

Exercise: Prove the result.

$$M_V(t) = M_{X+Y}(t) = \mathbb{E}\, e^{t(X+Y)} = \mathbb{E}\left[e^{tX} e^{tY}\right]$$

$$= \mathbb{E}\, e^{tX}\, \mathbb{E}\, e^{tY} = M_X(t)\, M_Y(t)$$

Exercise: Let X and Y be independent rvs such that

$$X \sim \text{Binomial}(n, p)$$
$$Y \sim \text{Binomial}(m, p)$$

Find the distribution of $U = X + Y$. $\sim$ Binom $(n+m, p)$

$$M_X(t) = [pe^t + (1-p)]^n$$

$$M_Y(t) = [pe^t + (1-p)]^m$$

mgf of Binom $(n+m, p)$

$$M_U(t) = M_X(t) M_Y(t) = [pe^t + (1-p)]^{n+m}$$

Exercise: Let X and Y be independent rvs such that

$$X \sim \text{Normal}(\mu_X, \sigma_X^2)$$
$$Y \sim \text{Normal}(\mu_Y, \sigma_Y^2)$$

Find the distribution of $U = X + Y$. $\sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

$*$

Exercise: Let X_1 and X_2 be independent rvs such that

$$X_1 \sim \text{Poisson}(\lambda_1)$$
$$X_2 \sim \text{Poisson}(\lambda_2)$$

Find the distribution of $Y = X_1 + X_2$. $\sim \text{Poisson}(\lambda_1 + \lambda_2)$

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