STAT 712 fa 2022 Lec 10 slides

Transformations of multiple random variables

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

We often wish to find the distribution of a function of two (or more) rvs:

Exercise: Let X_1 and X_2 be indep. Exponential(λ) rvs. Let $Y = X_1/(X_1 + X_2)$.

1. Find the cdf of Y.
2. Find the pdf of Y.

Theorem (*Bivariate transformation method*)

Let (X_1, X_2) *be a pair of cont. rvs with joint pdf* f_{X_1,X_2} *on* $\mathcal{X}$ *and*

$$Y_1 = g_1(X_1, X_2) \quad \text{and} \quad Y_2 = g_2(X_1, X_2),$$

where g_1 *and* g_2 *define a 1:1 transformation of* $\mathcal{X}$ *onto* $\mathcal{Y}$ *(define these).*

Let g_1^{-1} *and* g_2^{-1} *be the functions satisfying*

$$\begin{array}{l} y_1 = g_1(x_1, x_2) \\ y_2 = g_2(x_1, x_2) \end{array} \iff \begin{array}{l} x_1 = g_1^{-1}(y_1, y_2) \\ x_2 = g_2^{-1}(y_1, y_2) \end{array}$$

for all $(x_1, x_2) \in \mathcal{X}$ *and* $(y_1, y_2) \in \mathcal{Y}$.

Then the joint pdf of (Y_1, Y_2) *is given by*

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2))|J(y_1, y_2)|, \text{ for } (y_1, y_2) \in \mathcal{Y},$$

where $J(y_1, y_2)$ *is the Jacobian (next slide), if* $J(y_1, y_2)$ *is not always* 0 *on* $\mathcal{Y}$.

Jacobian

In the setup of the previous slide, the *Jacobian* of the transformation is defined as

$$J(y_1, y_2) = \left| \begin{array}{cc} \frac{\partial}{\partial y_1} g_1^{-1}(y_1, y_2) & \frac{\partial}{\partial y_2} g_1^{-1}(y_1, y_2) \\ \frac{\partial}{\partial y_1} g_2^{-1}(y_1, y_2) & \frac{\partial}{\partial y_2} g_2^{-1}(y_1, y_2) \end{array} \right|.$$

For real numbers a, b, c, d,

$$\left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = ad - bc.$$

This is called the *determinant*.

Exercise: Let X_1, X_2 be independent Normal$(0, 1)$ rvs.

1. Find the joint pdf of $Y_1 = X_1/X_2$ and $Y_2 = X_2$.
2. Find the marginal pdf of Y_1.

Exercise: Let X_1, X_2 have joint pdf

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\lambda^2} \exp\left[-\frac{x_1 + x_2}{\lambda}\right] \mathbf{1}(x_1 > 0, x_2 > 0).$$

1. Find the joint pdf of $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$.
2. Find the marginal pdf of Y_1.

Exercise: Let $X_1 \sim \text{Beta}(1,1)$ and $X_2 \sim \text{Beta}(2,1)$ be independent rvs.

1. Find the joint pdf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$.
2. Find the marginal pdf of Y_1.

Exercise: Let Z_1, Z_2 have the bivariate Normal distribution, with joint pdf

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right].$$

1. Find the joint pdf of $U_1 = Z_1 + Z_2$ and $U_2 = Z_1 - Z_2$.
2. Find the marginal pdfs of U_1 and U_2.

Exercise: Let Z_1, Z_2 have the bivariate Normal distribution, with joint pdf

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right].$$

1. Find the joint pdf of $U_1 = \min\{Z_1, Z_2\}$ and $U_2 = \max\{Z_1, Z_2\}$? (Not 1:1).
2. Find the marginal pdf of U_2 (work on this at home).

Now consider how $d \geq 1$ random variables behave together.

It is often convenient to organize our random variables into a random vector:

Random vector

A *random vector* is a vector in which each entry is a random variable.

Multivariate pdf/pmf

The *joint pdf* of $\mathbf{X} = (X_1, \dots, X_d)^T$ is the function $f : \mathbb{R}^d \to [0, \infty)$ such that

$$P(\mathbf{X} \in A) = \int_A f(\mathbf{x}) d\mathbf{x} \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d).$$

If $X_1, \dots, X_d$ are discrete, then the *joint pmf* of $\mathbf{X}$ is given by

$$p(\mathbf{x}) = P(\mathbf{X} = \mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d.$$

Multivariate transformation

Let $\mathbf{X} \in \mathbb{R}^d$ have joint pdf f with support $\mathcal{A}$. Consider the rvec

$$\mathbf{Y} = (g_1(\mathbf{X}), \dots, g_d(\mathbf{X}))^T$$

for some functions $g_1, \dots, g_d : \mathbb{R}^d \to \mathbb{R}$.

Moreover, given a partition $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_m$ of $\mathcal{A}$, where $P(\mathbf{X} \in \mathcal{A}_0) = 0$, suppose the transformation is $1:1$ on each of the sets $\mathcal{A}_1, \dots, \mathcal{A}_m$.

For $k = 1, \dots, m$, let the inverse transformation on $\mathcal{A}_k$ be given by

$$\mathbf{x} = (g_{1k}^{-1}(\mathbf{y}), \dots, g_{dk}^{-1}(\mathbf{y}))$$

for some functions $g_{1k}^{-1}, \dots, g_{dk}^{-1}$ and let $\mathbf{J}_k(\mathbf{y})$ be the corresponding Jacobian.

Then the joint pdf of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \sum_{k=1}^{m} f((g_{1k}^{-1}(\mathbf{y}), \dots, g_{dk}^{-1}(\mathbf{y}))^T)|\mathbf{J}_k(\mathbf{y})|.$$

Exercise: Let $Z_1, Z_2 \overset{\text{ind}}{\sim} \text{Normal}(0, 1)$ and consider the rvs

$$\begin{aligned} Y_1 &= Z_1^2 + Z_2^2 \\ Y_2 &= Z_1 / \sqrt{Z_1^2 + Z_2^2}. \end{aligned}$$

1. Find the joint pdf of (Y_1, Y_2).
2. Check whether Y_1 and Y_2 are independent.
3. Find the marginal pdf of Y_2.

Exercise: Let $X_1, X_2, X_3 \overset{\text{ind}}{\sim} \text{Exponential}(1)$ and consider

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2} \\ Y_2 &= \frac{X_1 + X_2}{X_1 + X_2 + X_3} \\ Y_3 &= X_1 + X_2 + X_3. \end{aligned}$$

1. Find the joint pdf of (Y_1, Y_2, Y_3).
2. Check whether Y_1, Y_2, and Y_3 are mutually independent.
3. Find the marginal distributions Y_1, Y_2, and Y_3.

1. Bivariate transformations
2. Multivariate transformations, including non-1:1
3. Sums of independent random variables

We extend our definition of independence to a collection of more than two rvs.

Mutual independence for a collection of rvs

Let the rvs $X_1, \dots, X_d$ have joint pdf/pmf $f(x_1, \dots, x_d)$ and marginal pdfs/pmfs such that $X_i \sim f_{X_i}$, $i = 1, \dots, d$. If

$$f(x_1, \dots, x_d) = \prod_{j=1}^d f_{X_j}(x_j) \quad \text{for all } (x_1, \dots, x_d) \in \mathbb{R}^d,$$

we say that $X_1, \dots, X_d$ are *mutually independent* rvs.

Theorem (Quick check for mutual independence)

The rvs $X_1, \dots, X_d$ with joint pdf/pmf f are mutually independent if and only if there exist functions $g_1, \dots, g_d$ such that

$$f(x_1, \dots, x_d) = \prod_{j=1}^d g_j(x_j) \quad \textit{for all} \quad (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Theorem (Expectation of a product of functions)

If $X_1, \dots, X_n$ are mutually independent, then

$$\mathbb{E}(g_1(X_1) \cdot \dots \cdot g_n(X_n)) = \textstyle\prod_{i=1}^n \mathbb{E} g_i(X_i)$$

for any functions $g_1, \dots, g_n : \mathbb{R} \to \mathbb{R}$.

Exercise: Prove the above.

Theorem (mgf method for sums of independent rvs)

Let $X_1, \ldots, X_n$ be independent rvs with mgfs $M_{X_1}, \ldots, M_{X_n}$, respectively. Then the mgf of $Y = X_1 + \cdots + X_n$ is given by

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Moreover, if $X_1, \ldots, X_n$ iid with mgf M_X then $M_Y(t) = [M_X(t)]^n$.

Exercise: Prove the above.

Exercise: Let $X_1, \ldots, X_n$ be ind. chi-squared rvs with dfs $\nu_1, \ldots, \nu_n$, resp. Find the distribution of $Y = X_1 + \cdots + X_n$.

Exercise: Let $X_1, \ldots, X_n$ be ind. Normal rvs with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1^2, \ldots, \sigma_n^2$, resp.

1. Find the distribution of $Y = X_1 + \cdots + X_n$.
2. Find the distribution of $V = a_1 X_1 + \cdots + a_n X_n$, for $a_1, \ldots, a_n \in \mathbb{R}$.
3. Find the distribution of $\bar{X}_n = (X_1 + \cdots + X_n)/n$.