STAT 712 fa 2022 Lec 14 slides

Convergence in probability, WLLN

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Reminder

For a sequence of real numbers $\{a_n\}_{n\geq 1}$, we say a_n *converges to* a as n goes to infinity if for every $\varepsilon > 0$ there exists an integer $N_\varepsilon \geq 1$ such that

$$|a_n - a| < \varepsilon \quad \text{for all } n \geq N_\varepsilon,$$

and we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ as $n \to \infty$.

Convergence in probability

Let $\{X_n\}_{n\geq 1}$ be a seq. of rvs. We say X_n *converges in probability* to a rv X if

$$\lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1 \quad \text{for every } \varepsilon > 0,$$

and we write $X_n \xrightarrow{\text{p}} X$.

Exercise: Let $Z_0, Z_1, Z_2, \ldots \overset{\text{ind}}{\sim} \text{Normal}(0,1)$ and set

$$X_i = Z_0 + Z_i \quad \text{for i} = 1, 2, \ldots$$

Show that $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n) \xrightarrow{\text{p}} Z_0$ as $n \to \infty$.

We are often interested in showing $X_n \xrightarrow{\text{p}} c$, where c is a constant, i.e.

$$\lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1 \quad \text{for every } \varepsilon > 0.$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, \theta)$. Show that $X_{(n)} \xrightarrow{\text{p}} \theta$.

If $\hat{\theta}_n$ is a rv with which we estimate θ, we call $\hat{\theta}_n$ a *consistent estimator* if $\hat{\theta}_n \xrightarrow{\text{p}} \theta$.

More about consistency next semester...

Theorem (Weak law of large numbers)

Let $X_1, \dots, X_n$ be iid with mean μ and variance $\sigma^2 < \infty$. Then $\bar{X}_n \xrightarrow{p} \mu$.

WLLN: Sample mean $\xrightarrow{\text{p}}$ population mean if population variance is finite.

There is also a *SLLN*, but this is an STAT 810/811 topic.

Exercise: Prove using Марков's inequality.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} F_X$ and set

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_i \leq x) \quad \text{for all } x \in \mathbb{R}.$$

Show that $\hat{F}_n(x) \xrightarrow{\text{p}} F_X(x)$ for each $x \in \mathbb{R}$.

Sufficient condition for convergence in probability to a constant

For a sequence $\{X_n\}_{n\geq 1}$ of rvs and a real number c,

$$\mathbb{E}X_n \to c \text{ and } \operatorname{Var} X_n \to 0 \quad \Longrightarrow \quad X_n \xrightarrow{\text{p}} c.$$

Exercise: Prove the above.

Exercise: Let $X_n = n^{-1} W_n$, where $W_n \sim \chi^2_n$ for $n = 1, 2, \dots$ Show that $X_n \xrightarrow{\text{p}} 1$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x; \beta) = \beta x^{-(\beta+1)} \mathbf{1}(x \geq 1)$ and let

$$\hat{\beta}_n = \frac{n-1}{\sum_{i=1}^n \log X_i}$$

Check whether $\hat{\beta}_n$ converges to β by analyzing $\mathbb{E}\hat{\beta}_n$ and $\text{Var}\,\hat{\beta}_n$.

Theorem (Helper results for proving convergence in probability)

Let $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} Y$. *Then*

1. $cX_n \xrightarrow{p} cX$ *for* $c \neq 0$.
2. $X_n \pm Y_n \xrightarrow{p} X \pm Y$
3. $X_n Y_n \xrightarrow{p} XY$
4. $X_n / Y_n \xrightarrow{p} X/Y$, *provided* $P(Y = 0) = 0$.
5. *For any continuous function* g, $g(X_n) \xrightarrow{p} g(X)$ *(continuous mapping).*
6. *For any sequences* $\{a_n\}_{n \geq 1}$, $\{b_n\}_{n \geq 1}$ *s.t.* $\lim_{n \to \infty} a_n = 1$ *and* $\lim_{n \to \infty} b_n \to 0$,

$$a_n X_n + b_n \xrightarrow{p} X.$$

Exercise: Prove 1, 2, and 5 from above.

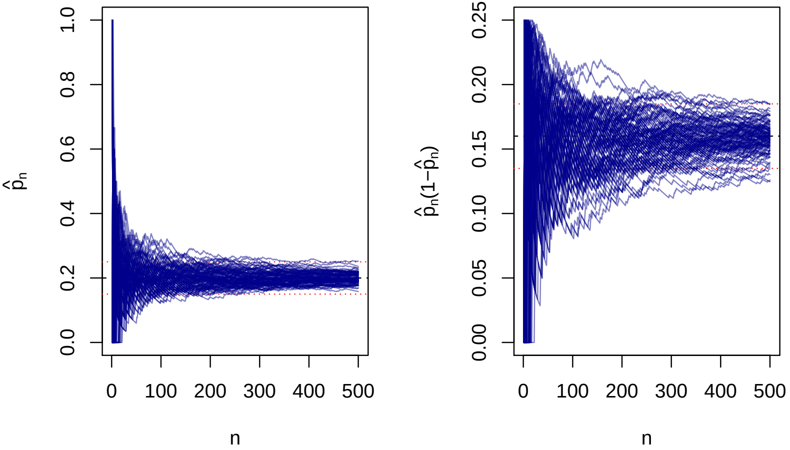
Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$. Let $Y = X_1 + \dots + X_n$ and consider

$$\hat{p}_n = \frac{Y}{n} \quad \text{and} \quad \tilde{p}_n = \frac{Y+2}{n+4}.$$

Check convergence to p of $\hat{p}_n$ and $\tilde{p}_n$.

Exercise: Let $X_1, \ldots, X_n \in \text{Bernoulli}(p)$, $\hat{p}_n = \bar{X}_n$. Argue that

$$\hat{p}_n(1 - \hat{p}_n) \xrightarrow{\text{p}} p(1 - p) \quad \text{as } n \to \infty.$$



Exercise: Let $\{(X_n, Y_n)\}_{n\geq 1}$ be iid rv pairs with $\text{corr}(X_1, Y_1) = \rho$. Show that

$$\hat{\rho}_n = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Y_i - \bar{Y}_n)^2}} \xrightarrow{\text{p}} \rho,$$

provided $\mathbb{E}X_1^4$ and $\mathbb{E}Y_1^4$ are finite.

Approaches for establishing convergence in probability to a constant

1. Directly show $\lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1$ for all $\varepsilon > 0$.
2. Show $\mathbb{E}X_n \to c$ and $\operatorname{Var} X_n \to 0$ as $n \to \infty$.
3. Use the WLLN if applicable in conjunction with the helper results.

Exercises: Let $X_1, \dots, X_n$ be a rs. Consider showing $S_n^2 \xrightarrow{\text{p}} \sigma^2$ in above ways.