STAT 712 fa 2022 Lec 14 slides

Convergence in probability, WLLN

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Reminder

For a sequence of real numbers $\{a_n\}_{n\geq 1}$, we say a_n *converges to* a as n goes to infinity if for every $\varepsilon > 0$ there exists an integer $N_\varepsilon \geq 1$ such that

$$|a_n - a| < \varepsilon \quad \text{for all } n \geq N_\varepsilon,$$

and we write $\lim_{n\to\infty} a_n = a$ or $\boxed{a_n \to a}$ as $n \to \infty$.

$X_n \xrightarrow{p} X$ as $n \to \infty$.

Convergence in probability

Let $\{X_n\}_{n\geq 1}$ be a seq. of rvs. We say X_n *converges in probability* to a rv X if

$$\lim_{n\to\infty} \underbrace{\boxed{P(|X_n - X| < \varepsilon)}}_{a_n} = \underset{a}{1} \quad \text{for every } \varepsilon > 0,$$

X_n

$X-\varepsilon \quad X \quad X+\varepsilon$

and we write $X_n \xrightarrow{\text{p}} X$.

Exercise: Let $Z_0, Z_1, Z_2, \ldots \overset{\text{ind}}{\sim} \text{Normal}(0,1)$ and set

$$X_i = Z_0 + Z_i \quad \text{for i} = 1, 2, \ldots$$

Show that $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n) \xrightarrow{\text{p}} Z_0$ as $n \to \infty$. ✓

We are often interested in showing $X_n \xrightarrow{\text{p}} c$, where c is a constant, i.e.

$$\lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1 \quad \text{for every } \varepsilon > 0.$$

Choose $\varepsilon > 0$.

$$P\left(\left|\bar{X}_n - Z_0\right| < \varepsilon\right)$$

$$= P\left(\left|\frac{1}{n}\left(X_1 + \cdots + X_n\right) - Z_0\right| < \varepsilon\right)$$

$$= P\left(\left|\frac{1}{n}\left(Z_0 + Z_1 + Z_0 + Z_2 + \cdots + Z_0 + Z_n\right) - Z_0\right| < \varepsilon\right)$$

$$= P\left(\left|\frac{1}{n}\left(Z_1 + \cdots + Z_n\right)\right| < \varepsilon\right)$$

$$= P\left(\left|\bar{Z}_n\right| < \varepsilon\right), \quad \bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$$

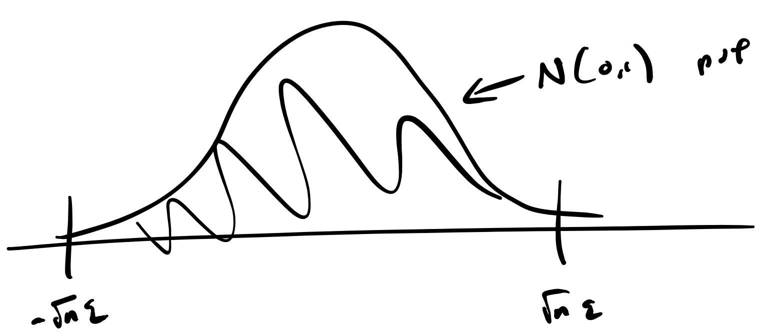
$$= P\left(\left|\sqrt{n}\,\bar{Z}_n\right| < \sqrt{n}\,\varepsilon\right) \quad \bar{Z}_n \sim N\left(0, \frac{1}{n}\right)$$

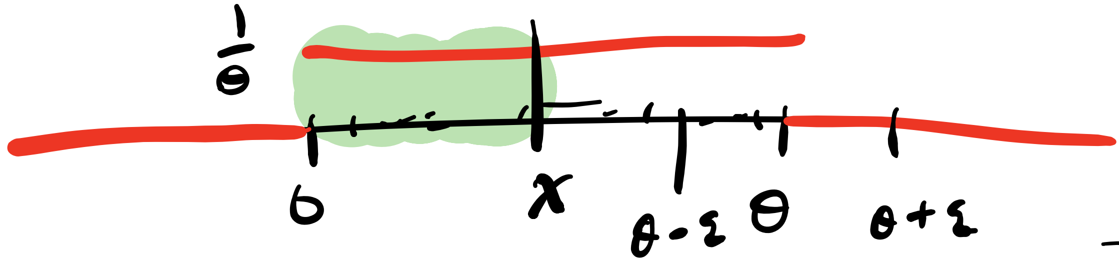
$$= P\left(|Z| < \underbrace{\sqrt{n}\,\varepsilon}\right), \quad Z \sim N(0,1)$$

$$= P\left(-\sqrt{n}\,\varepsilon < Z < \sqrt{n}\,\varepsilon\right)$$

$$= \underbrace{\Phi(\sqrt{n}\,\varepsilon)}_{\to 1 \text{ as } n \to \infty} - \underbrace{\Phi(-\sqrt{n}\,\varepsilon)}_{\to 0 \text{ as } n \to \infty}$$

$$\to 1 \quad \text{as } n \to \infty.$$





Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, \theta)$. Show that $X_{(n)} \xrightarrow{\text{p}} \theta$.

If $\hat{\theta}_n$ is a rv with which we estimate θ, we call $\hat{\theta}_n$ a *consistent estimator* if $\hat{\theta}_n \xrightarrow{\text{p}} \theta$.

More about consistency next semester...

$X_1, \dots, X_n \overset{iid}{\sim} \text{Unif}(0, \theta)$,

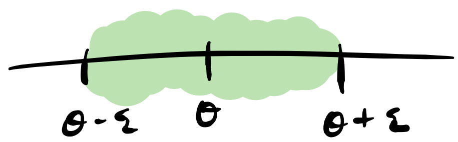
$f_X(x) = \frac{1}{\theta} \mathbb{1}(0 < x < \theta)$

$F_X(x) = \frac{x}{\theta}$ for $0 < x < \theta$.

$$f_{X_{(n)}}(x) = n \left[F_X(x)\right]^{n-1} f_X(x)$$

$$= n \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} \mathbb{1}(0 < x < \theta)$$

Now (choose $\varepsilon > 0$)



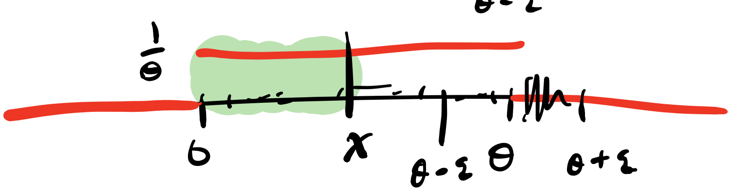
$P\left(|X_{(n)} - \theta| < \varepsilon \right)$ as $n \to \infty$.

$$= P\left(\theta - \varepsilon < X_{(n)} < \theta + \varepsilon\right)$$

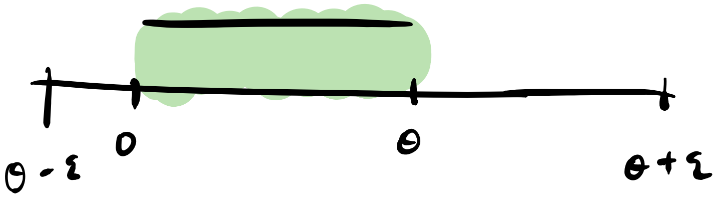
$$= \int_{\theta-\varepsilon}^{\theta+\varepsilon} n \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} \mathbb{1}(0 < x < \theta)\, dx$$

$$= \int_{\theta-\varepsilon}^{\theta} n \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx$$

$$= \frac{n}{\theta^n} \int_{\theta-\varepsilon}^{\theta} x^{n-1}\, dx = \frac{n}{\theta^n} \left[\frac{x^n}{n}\right]\Bigg|_{\theta-\varepsilon}^{\theta}$$



$$= \frac{\theta^n - (\theta - \varepsilon)^n}{\theta^n}$$



$$= 1 - \underbrace{\left(\frac{\theta - \varepsilon}{\theta}\right)^n}_{\to 0 \text{ as } \to \infty}$$

$$\to 1 \quad \text{as} \quad n \to \infty.$$

Theorem (Weak law of large numbers)

Let $X_1, \ldots, X_n$ be iid with mean μ and $\boxed{\textit{variance } \sigma^2 < \infty.}$ *Then $\bar{X}_n \xrightarrow{p} \mu$.*

WLLN: Sample mean $\xrightarrow{p}$ population mean if population variance is finite.

There is also a *SLLN*, but this is an STAT 810/811 topic.

Exercise: Prove using Марков's inequality

for nonneg rv W, $P(W > a) \le \frac{\mathbb{E}W}{a}$

Choose $\varepsilon > 0$.

$$P(|\bar{X}_n - \mu| < \varepsilon) = 1 - P(|\bar{X}_n - \mu| > \varepsilon)$$
$$= 1 - \underbrace{P(|\bar{X}_n - \mu|^2 > \varepsilon^2)}$$

$$L \leq \frac{\mathbb{E}|\bar{X}_n - \mu|^2}{\varepsilon^2}$$

$$\geq 1 - \frac{\mathbb{E}|\bar{X}_n - \mu|^2}{\varepsilon^2}$$

$$= 1 - \frac{\sigma^2}{n\varepsilon^2}$$

$$\longrightarrow 1 \quad \text{as} \quad n \to \infty$$

Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} F_X$ and set "Empirical distribution function"

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \leq x) \quad \text{for all } x \in \mathbb{R}.$$

" $\dfrac{\#\{X_i \leq x\}}{n}$ "

Show that $\hat{F}_n(x) \xrightarrow{p} F_X(x)$ for each $x \in \mathbb{R}$.

Let $Y_i = \mathbb{1}(X_i \leq x) \sim \text{Bernoulli}(F_X(x))$

$\mathbb{E} Y_i = F_X(x)$

$\text{Var } Y_i = F_X(x)[1 - F_X(x)] < \infty.$

$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}_n \xrightarrow{p} F_X(x)$

Sufficient condition for convergence in probability to a constant

For a sequence $\{X_n\}_{n\geq 1}$ of rvs and a real number c,

$$\mathbb{E}X_n \to c \text{ and } \operatorname{Var} X_n \to 0 \implies X_n \xrightarrow{\text{p}} c.$$

Exercise: Prove the above.

$$P(|X_n - c| < \varepsilon) = 1 - P(|X_n - c| \geq \varepsilon)$$
$$= 1 - P(|X_n - c|^2 \geq \varepsilon^2)$$
$$\geq 1 - \frac{\mathbb{E}|X_n - c|^2}{\varepsilon^2} \overset{?}{\to} 0$$

$$\mathbb{E}\left|X_n - c\right|^2 = \mathbb{E}\Big((X_n - \mathbb{E}X_n) + (\mathbb{E}X_n - c) \Big)^2$$

$$= \mathbb{E}\,(X_n - \mathbb{E}X_n)^2 +$$

$$+ \boxed{2\,\mathbb{E}\,(X_n - \mathbb{E}X_n)(\mathbb{E}X_n - c)}^{=0}$$

$$+ \mathbb{E}\,(\mathbb{E}X_n - c)^2$$

$$= \underbrace{\operatorname{Var} X_n}_{\to 0} + \overbrace{\underbrace{(\mathbb{E}X_n - c)^2}_{\to 0}}^{(\text{"bias"})^2}$$

$$\longrightarrow 0$$

Exercise: Let $X_n = n^{-1}W_n$, where $W_n \sim \chi^2_n$ for $n = 1, 2, \ldots$ Show that $X_n \xrightarrow{p} 1$.

$$P(|X_n - 1| < \varepsilon) = P(|n^{-1}W_n - 1| < \varepsilon)$$
$$= P(|W_n - n| < n\varepsilon)$$
$$= P(n - n\varepsilon \leq W_n \leq n + n\varepsilon)$$
$$= P(n(1-\varepsilon) \leq W_n \leq n(1+\varepsilon))$$

$$\chi^2_n = \text{Gamma}\left(\frac{n}{2}, 2\right) = \int_{n(1-\varepsilon)}^{n(1+\varepsilon)} \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} w^{n/2 - 1} e^{-w/2}\, dw$$

$\alpha \quad \beta^2$

not nice

<u>Instead</u>:

$$\mathbb{E} X_n = \mathbb{E}\left[\frac{1}{n} W_n\right] = \frac{1}{n} \mathbb{E} W_n = \frac{1}{n} \cdot n = 1$$

$$\text{Var } X_n = \text{Var}\left[\frac{1}{n} W_n\right] = \frac{1}{n^2} \text{Var } W_n = \frac{1}{n^2} 2n = \frac{2}{n} \to 0$$

as $n \to \infty$

$$\Rightarrow \quad X_n \xrightarrow{p} 1.$$

Exercise: Let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} f_X(x; \beta) = \beta x^{-(\beta+1)} \mathbf{1}(x \geq 1)$ and let

$$\hat{\beta}_n = \frac{n-1}{\sum_{i=1}^n \log X_i}$$

Check whether $\hat{\beta}_n$ converges to β by analyzing $\mathbb{E}\hat{\beta}_n$ and $\operatorname{Var}\hat{\beta}_n$.

(i) get dist of $\log X_1$

(ii) get dist of $\sum_{i=1}^{n} \log X_i$

Theorem (Helper results for proving convergence in probability)

Let $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} Y$. *Then*

1. $cX_n \xrightarrow{p} cX$ *for* $c \neq 0$.
2. $X_n \pm Y_n \xrightarrow{p} X \pm Y$
3. $X_nY_n \xrightarrow{p} XY$
4. $X_n/Y_n \xrightarrow{p} X/Y$, *provided* $P(Y = 0) = 0$.
5. *For any continuous function* g, $g(X_n) \xrightarrow{p} g(X)$ *(continuous mapping).*
6. *For any sequences* $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ *s.t.* $\lim_{n\to\infty} a_n = 1$ *and* $\lim_{n\to\infty} b_n \to 0$,

$$a_nX_n + b_n \xrightarrow{p} X.$$

Exercise: Prove 1, 2, [redacted] from above.

(1) Suppose $X_n \xrightarrow{P} X$. Then

$$P\left(|cX_n - cX| < \varepsilon \right)$$

$|ab| = |a||b|$

$$= P\left(|c|\,|X_n - X| < \varepsilon \right)$$

$$= P\left(|X_n - X| < \underbrace{\frac{\varepsilon}{|c|}}_{\varepsilon^*} \right)$$

$$\to 1 \quad \text{as} \quad n \to \infty.$$

(2) If $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y$, then

$$P\left(|(X_n + Y_n) - (X + Y)| < \varepsilon \right)$$

$P(A \cup B) \leq P(A) + P(B)$

$$= P\left(|(X_n - X) + (Y_n - Y)| < \varepsilon \right)$$

$$\geq P\left(|X_n - X| < \frac{\varepsilon}{2} \cap |Y_n - Y| < \frac{\varepsilon}{2} \right)$$

$$= 1 - P\left(|X_n - X| \geq \frac{\varepsilon}{2} \cup |Y_n - Y| \geq \frac{\varepsilon}{2} \right)$$

$$\geq 1 - P\left(|X_n - X| \geq \frac{\varepsilon}{2} \right) - P\left(|Y_n - Y| \geq \frac{\varepsilon}{2} \right)$$

$$= 1 - \left(1 - P\left(|X_n - X| < \frac{\varepsilon}{2} \right)\right) - \left(1 - P\left(|Y_n - Y| < \frac{\varepsilon}{2} \right)\right)$$

$$= P\left(|X_n - X| < \frac{\varepsilon}{2} \right) + P\left(|Y_n - Y| < \frac{\varepsilon}{2} \right) - 1$$

$$\underbrace{\qquad\qquad}_{\to 1} \qquad \underbrace{\qquad\qquad}_{\to 1} \quad \text{as } n \to \infty$$

$$\to 1 \quad \text{as } n \to \infty. \qquad \}$$

$$|a+b| \le |a| + |b| \qquad\qquad (A \cap B)^c = A^c \cup B^c$$

$$\underbrace{\left\{ |X_n - X| < \frac{\varepsilon}{2} \cap |Y_n - Y| < \frac{\varepsilon}{2} \right\}}_{A} \Rightarrow \underbrace{\left\{ \left| (X_n - X) + (Y_n - Y) \right| < \varepsilon \right.}_{B}$$

$A \Rightarrow B$ means $A \subset B$, so $P(A) \le P(B)$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$. Let $Y = X_1 + \cdots + X_n$ and consider

just $\bar{X}_n$

$$\hat{p}_n = \frac{Y}{n} \quad \text{and} \quad \tilde{p}_n = \frac{Y+2}{n+4}.$$

Check convergence to p of $\hat{p}_n$ and $\tilde{p}_n$.

$\hat{p}_n \xrightarrow{p} p$ from WLLN.

$$\tilde{p}_n = \frac{Y+2}{n+4} = \frac{(Y+2)}{n}\frac{n}{n+4} = \boxed{\frac{Y}{n}\underbrace{\left(\frac{n}{n+4}\right)}_{\to 1} + \underbrace{\frac{2}{n+4}}_{\to 0}} \xrightarrow{p} p$$

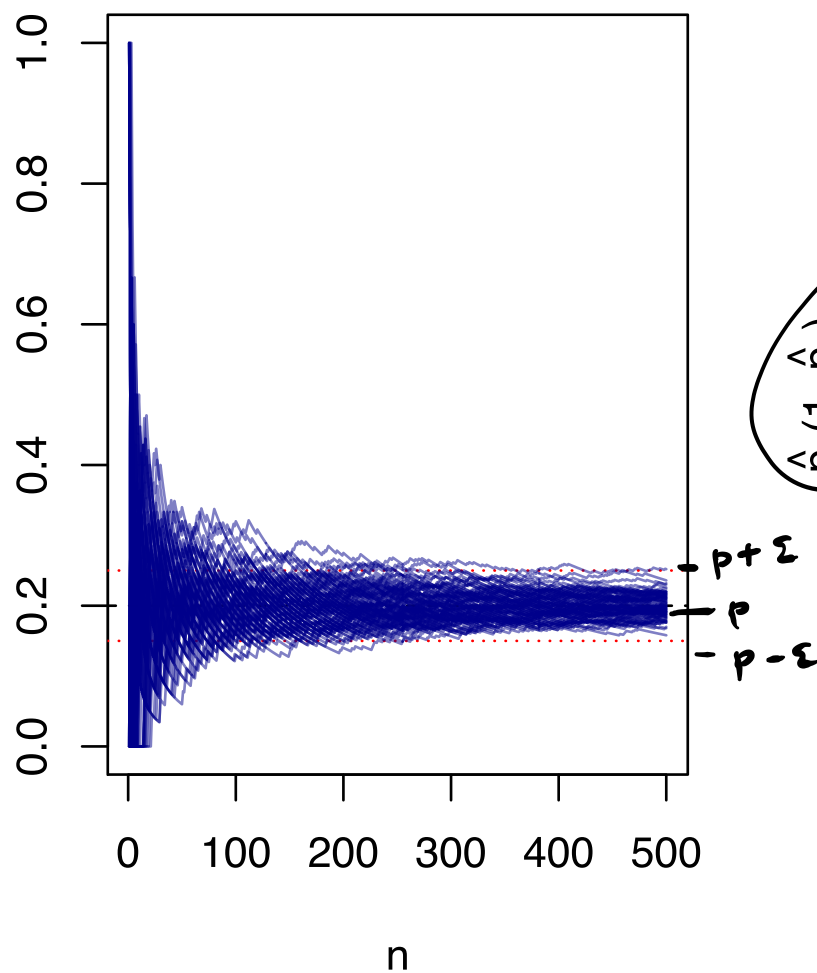
$\frac{Y}{n} \xrightarrow{p} p$ by WLLN

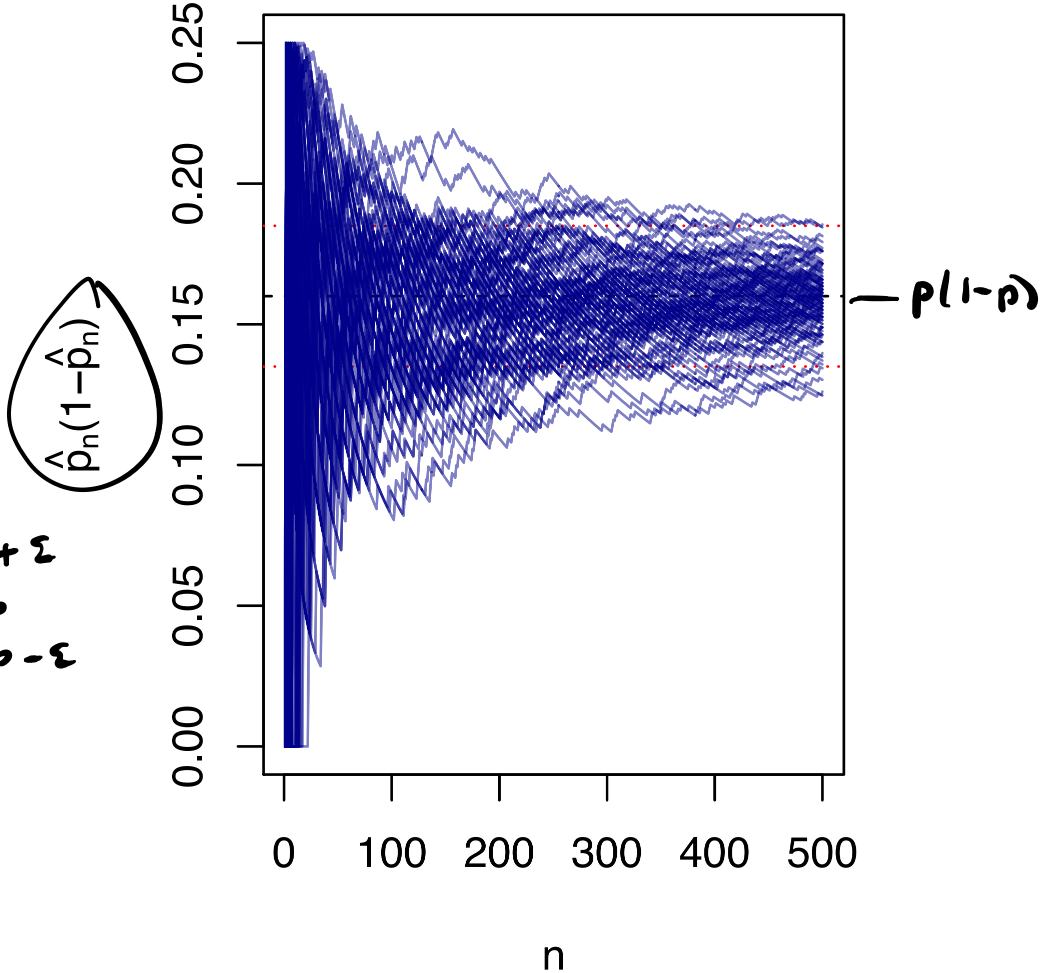
Exercise: Let $X_1, \dots, X_n \overset{iid}{\sim}$ Bernoulli(p), $\hat{p}_n = \bar{X}_n$. Argue that

$$\hat{p}_n(1-\hat{p}_n) \xrightarrow{p} p(1-p) \quad \text{as } n \to \infty.$$

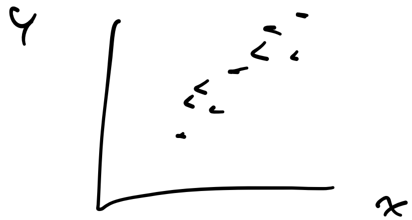
$\hat{p}_n \xrightarrow{p} p$ by WLLN $\quad g(x) = x(1-x)$, continuous.

$p = 0.2$





$X_n \xrightarrow{p} c$ if $\lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1 \quad \forall\, \varepsilon > 0.$



$= \rho$ rho

Exercise: Let $\{(X_n, Y_n)\}_{n\geq 1}$ be iid rv pairs with $\text{corr}(X_1, Y_1) = \rho$. Show that

$$\boxed{\hat{\rho}_n = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Y_i - \bar{Y}_n)^2}}} \xrightarrow{p} \rho,$$

provided $\mathbb{E}X_1^4$ and $\mathbb{E}Y_1^4$ are finite.

$$\rho = g(\mathbb{E}X_1, \mathbb{E}Y_1, \mathbb{E}X_1Y_1, \mathbb{E}X_1^2, \mathbb{E}Y_1^2)$$

$p \uparrow \quad p \uparrow \quad p \uparrow \quad p \uparrow \quad p \uparrow$

$$\hat{\rho}_n = g\left(\bar{X}_n, \bar{Y}_n, \frac{1}{n}\sum X_iY_i, \frac{1}{n}\sum X_i^2, \frac{1}{n}\sum Y_i^2\right)$$

$$\hat{\rho}_n = \frac{\sum_{i=1}^{n} X_i Y_i - n \bar{X}_n \bar{Y}_n}{\sqrt{\left(\sum_{i=1}^{n} X_i^2 - n \bar{X}_n^2\right)\left(\sum_{i=1}^{n} Y_i^2 - n \bar{Y}_n^2\right)}}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \bar{X}_n \bar{Y}_n}{\sqrt{\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \bar{Y}_n^2\right)}}$$

$$\bar{X}_n \xrightarrow{p} \mathbb{E}X_1$$
$$\bar{Y}_n \xrightarrow{p} \mathbb{E}Y_1$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i Y_i = \frac{1}{n}\sum_{i=1}^{n} U_i, \quad U_i = X_i Y_i$$

$$\mathbb{E}U_i = \mathbb{E}X_1 Y_1$$

$$\mathrm{Var}\, U_i = \mathrm{Var}(X_1 Y_1)$$

$$\frac{1}{n}\sum_{i=1}^{n} U_i \xrightarrow{p} \mathbb{E}X_1 Y_1 \quad \text{as } n \to \infty \text{ if } \mathrm{Var}(X_1 Y_1) < \infty.$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to \mathbb{E}X_1^2 \quad \text{as } n \to \infty \text{ if } \underline{\mathrm{Var}\, X_1^2} < \infty.$$

$$\frac{1}{n}\sum_{i=1}^{n} Y_i^2 \to \mathbb{E}Y_1^2 \quad \text{as } n \to \infty \text{ if } \mathrm{Var}\, Y_i^2 < \infty.$$

$$\rho(X_1, Y_1) = \frac{\mathrm{Cov}(X_1, Y_1)}{\sqrt{\mathrm{Var}\, X_1 \, \mathrm{Var}\, Y_1}} = \frac{\mathbb{E}X_1 Y_1 - \mathbb{E}X_1 \, \mathbb{E}Y_1}{\sqrt{\left(\mathbb{E}X_1^2 - (\mathbb{E}X_1)^2\right)\left(\mathbb{E}Y_1^2 - (\mathbb{E}Y_1)^2\right)}}$$

$X_n \xrightarrow{p} c$:

Approaches for establishing convergence in probability to a constant

1. Directly show $\lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1$ for all $\varepsilon > 0$.
2. Show $\mathbb{E}X_n \to c$ ✓ and $\text{Var}\, X_n \to 0$ as $n \to \infty$.
3. Use the WLLN if applicable in conjunction with the helper results.

Exercises: Let $X_1, \ldots, X_n$ be a rs. Consider showing $S_n^2 \xrightarrow{p} \sigma^2$ in above ways.

(1) For $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(|S_n^2 - \sigma^2| < \varepsilon\right),$$ this is hard,

depends on knowing dist. of S_n^2.

(2) $\mathbb{E} S_n^2 = \sigma^2$

$$\operatorname{Var} S_n^2 = \frac{1}{n}\Big(\theta_4 - \frac{n-3}{n-1}\underbrace{\theta_2^2}_{(\sigma^2)^2}\Big), \quad \theta_4 = \mathbb{E}(X-\mathbb{E}X)^4, \quad \theta_2 = \mathbb{E}(X-\mathbb{E}X)^2$$

$$\longrightarrow 0$$

as $n \to \infty$.

So $S_n^2 \xrightarrow{p} \sigma^2$ if $\theta_4 < \infty$.

(3)

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}_n^2\right)$$

$$= \overbrace{\frac{n}{n-1}}^{\to 1}\ \underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i^2}_{\xrightarrow{p}\ \mathbb{E}X^2} - \overbrace{\frac{n}{n-1}}^{\to 1}\ \underbrace{\bar{X}_n}_{\xrightarrow{p}\ \mathbb{E}X}{}^2$$

$\bar{X}_n \xrightarrow{p} \mathbb{E}X$ by WLLN

$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} \mathbb{E}X^2$ by WLLN provided $\operatorname{Var}(X^2) < \infty$

$$\xrightarrow{p} \mathbb{E}X^2 - (\mathbb{E}X)^2 = \operatorname{Var} X = \sigma^2$$