

# STAT 712 fa 2022 Lec 15 slides

## Central limit theorem, Slutsky's theorem, delta method

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

- 1 Convergence in distribution
- 2 Central limit, continuous mapping, and Slutsky's theorems
- 3 Delta method

## Recall:

If  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ , then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1), \quad \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}, \quad \frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2$$



What if  $X_1, \dots, X_n$  are not sampled from a Normal distribution?

## Central limit theorem informally stated

If  $X_1, \dots, X_n$  are iid with mean  $\mu$  and variance  $\sigma^2 < \infty$ , then

$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$  behaves more and more like  $Z \sim N(0, 1)$  for larger and larger  $n$ .

Same is true of  $\frac{\bar{X}_n - \mu}{\hat{\sigma}_n/\sqrt{n}}$ , where  $\hat{\sigma}_n \xrightarrow{p} \sigma$ .

Can use to build large- $n$  CI for  $\mu$  when population non-Normal.

## Convergence in distribution

Seq. of rvs  $Y_1, Y_2, \dots$  w/cdfs  $F_{Y_1}, F_{Y_2}, \dots$  *converges in distribution* to  $Y \sim F_Y$  if

$$\lim_{n \rightarrow \infty} F_{Y_n}(y) = F_Y(y),$$

for all  $y \in \mathbb{R}$  at which  $F_Y(y)$  is continuous. We write  $Y_n \xrightarrow{D} Y$ .

Formalizes “behaves more and more like”.

Refer to dist. with cdf  $F_Y$  as the *asymptotic distribution* or *limiting distribution*.

## Theorem (Convergence in prob. implies convergence in dist.)

For any  $\{X_n\}_{n \geq 1}$  and  $X$  we have  $X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$ . (See ex. 5.40 of CB)

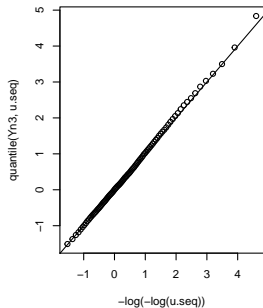
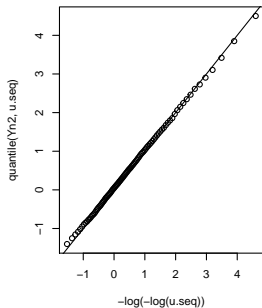
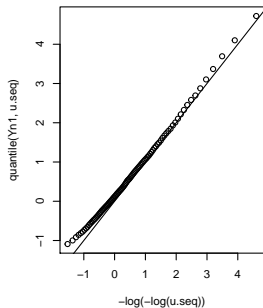
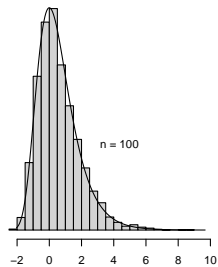
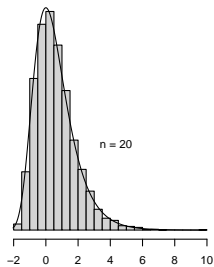
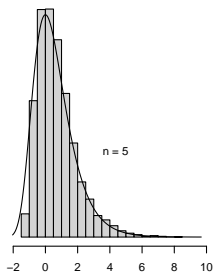
**Exercise:** For  $n \geq 1$ , let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Exponential}(\lambda)$  and let

$$Y_n = (X_{(n)} - \lambda \log n) / \lambda \quad \text{and} \quad Y \sim F_Y(y) = e^{-e^{-y}} \text{ for } y \in \mathbb{R}.$$

Show that  $Y_n \xrightarrow{D} Y$ .



$Y_n$  is an *asymptotic pivot quantity*: limiting dist. known and parameter-free.



## Theorem (Convergence in distribution to a constant)

For a sequence of rvs  $\{X_n\}_{n \geq 1}$ , the following statements are equivalent

- 1  $\lim_{n \rightarrow \infty} P(|X_n - \mu| > \varepsilon) = 0$
- 2  $\lim_{n \rightarrow \infty} P(X_n \leq x) = \begin{cases} 0 & x < \mu \\ 1 & x > \mu \end{cases}$

(See ex. 5.41 in CB)

So  $X_n \xrightarrow{P} \mu \iff X_n \xrightarrow{D} X$ , where  $X$  is a rv equal to  $\mu$  with probability 1.

**Discuss:** The second statement has  $x > \mu$  instead of  $x \geq \mu$ . Important?

A distribution which puts all its mass on a single value is a *degenerate* distribution.



- 1 Convergence in distribution
- 2 Central limit, continuous mapping, and Slutsky's theorems
- 3 Delta method

## Theorem (Central limit theorem)

If  $X_1, \dots, X_n$  are i.i.d. from a dist. with mgf defined in a neighborhood of zero, then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} Z \sim \text{Normal}(0, 1) \text{ as } n \rightarrow \infty,$$

where  $\mu$  and  $\sigma$  are the population mean and standard deviation.

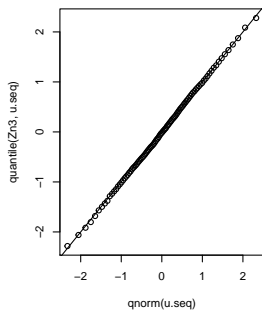
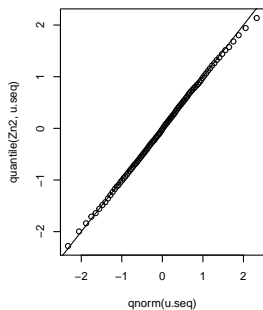
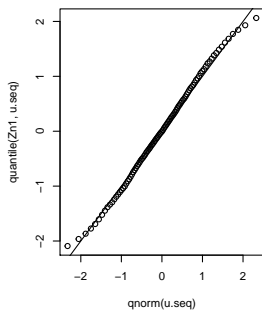
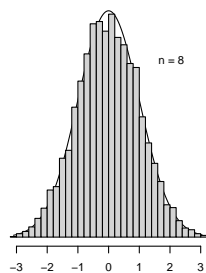
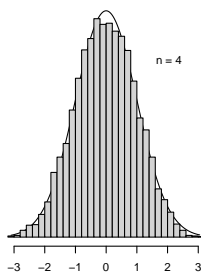
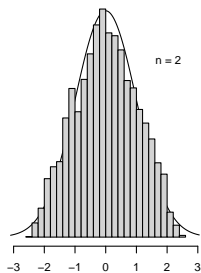
Mgf assumption not needed. Can prove CLT requiring only  $\sigma^2 < \infty$  (advanced).



**Exercise:** Prove using mgfs.

**Exercise:** Let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Uniform}(a, b)$ .

- 1 Give a function of  $\bar{X}_n$  which is asymptotically standard Normal.
- 2 Run some simulations to investigate the rate of convergence.



## Theorem (Continuous mapping theorem)

If  $X_n \xrightarrow{D} X$  then  $g(X_n) \xrightarrow{D} g(X)$  for any continuous function  $g$ .

### Examples:

- ❶ Let  $X_1, \dots, X_n$  be iid with mean  $\mu$  and variance  $\sigma^2 < \infty$ .

$$\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{D} Z \sim \text{Normal}(0, 1) \implies [\sqrt{n}(\bar{X}_n - \mu)/\sigma]^2 \xrightarrow{D} Z^2 \sim \chi_1^2$$

- ❷ Let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Exponential}(\lambda)$  and  $Y_n = (X_{(n)} - \lambda \log n)/\lambda$

$$Y_n \xrightarrow{D} Y \sim F_Y(y) = e^{-e^{-y}} \implies e^{-Y_n} \xrightarrow{D} e^{-Y} \sim \text{Exponential}(1)$$

## Theorem (Slutsky's theorem)

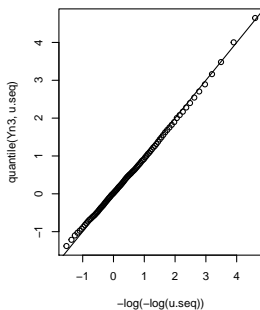
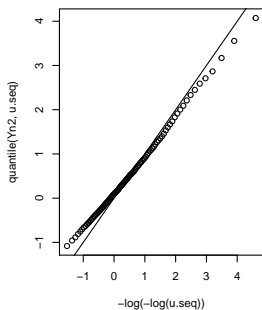
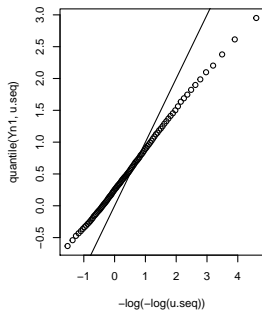
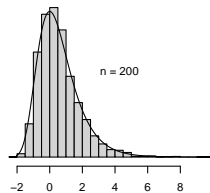
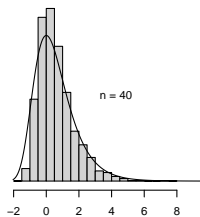
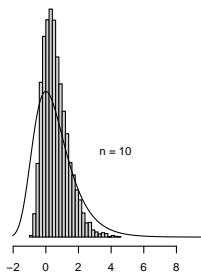
If  $X_n \xrightarrow{D} X$  and  $Y_n \xrightarrow{P} a$ , then

- 1  $X_n + Y_n \xrightarrow{D} X + a$
- 2  $X_n Y_n \xrightarrow{D} Xa$
- 3  $X_n/Y_n \xrightarrow{D} X/a$  provided  $a \neq 0$

**Exercise:** For  $n \geq 1$ , let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Exponential}(\lambda)$  and let

$$Y_n = (X_{(n)} - \lambda \log n)/\lambda \quad \text{and} \quad Y \sim F_Y(y) = e^{-e^{-y}} \text{ for } y \in \mathbb{R}.$$

Consider whether  $\tilde{Y}_n = (X_{(n)} - \bar{X}_n \log n)/\bar{X}_n$  converges in distribution to  $Y$ .



## Theorem (Corollary of Slutsky's theorem and central limit theorem)

Let  $X_1, \dots, X_n$  be a rs from a dist. with mean  $\mu$  and variance  $\sigma^2 < \infty$ . Then

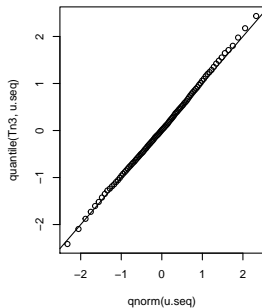
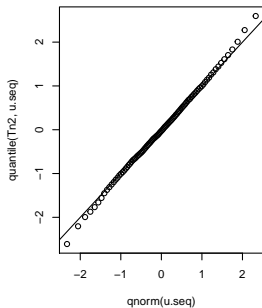
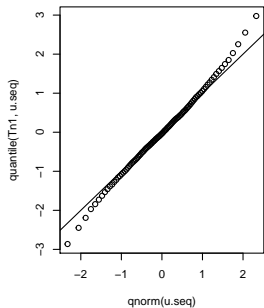
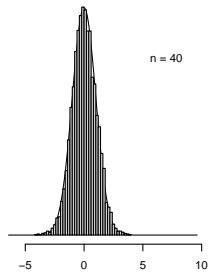
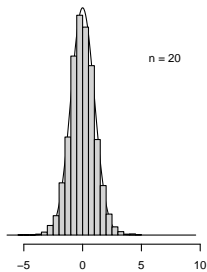
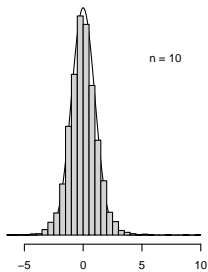
$$\frac{\bar{X}_n - \mu}{\hat{\sigma}_n / \sqrt{n}} \xrightarrow{D} Z \sim \text{Normal}(0, 1) \text{ as } n \rightarrow \infty,$$

provided  $\hat{\sigma}_n \xrightarrow{P} \sigma$ .

### Exercise:

- 1 Show how the CLT and Slutsky's theorem imply the above.
- 2 Illustrate this result in simulation for  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Uniform}(a, b)$ .





**Exercises:** Derive asymptotic confidence intervals for

- 1  $\mu$  based on  $X_1, \dots, X_n$  iid with mean  $\mu$  and finite 4th moment.
- 2  $p$  based on  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Bernoulli}(p)$ .

**Exercise:** For  $n = 10, 20, 40, 80, 160, 320$ , simulate coverage of

$$\bar{X}_n \pm z_{0.025} \cdot S_n / \sqrt{n}$$

for mean of  $\text{Gamma}(\alpha = 2, \beta = 4)$  distribution.

$n$	10	20	40	80	160	320
coverage	0.887	0.925	0.934	0.938	0.947	0.947

The *coverage* of a CI is the probability with which it contains its target.

- 1 Convergence in distribution
- 2 Central limit, continuous mapping, and Slutsky's theorems
- 3 Delta method**

## Theorem (Delta method)

Let  $Y_n$  be a sequence of rvs and  $\theta$  be a real number such that

$$\sqrt{n}(Y_n - \theta) \xrightarrow{D} \text{Normal}(0, \sigma^2) \quad \text{as } n \rightarrow \infty.$$

Then for any function  $g$  such that  $g'(\theta)$  exists and is not equal to zero, we have

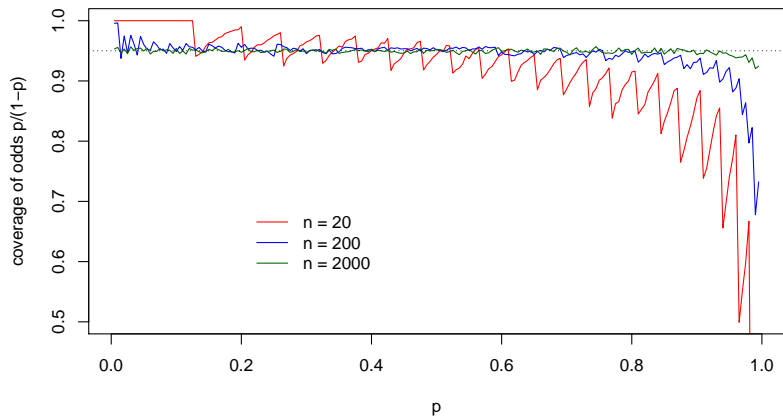
$$\sqrt{n}[g(Y_n) - g(\theta)] \xrightarrow{D} \text{Normal}(0, [g'(\theta)]^2 \sigma^2) \quad \text{as } n \rightarrow \infty.$$

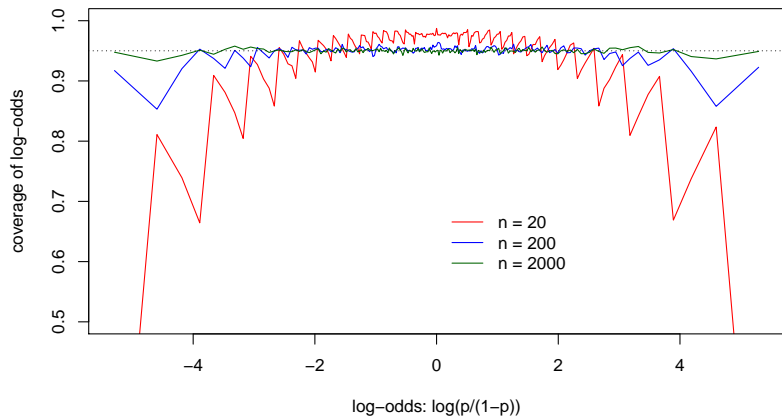
**Exercise:** Prove using Taylor's theorem, which gives

$$g(y) = g(\theta) + g'(\theta)(y - \theta) + R(y, \theta), \quad \text{with } \lim_{y \rightarrow \theta} \frac{R(y, \theta)}{(y - \theta)} = 0.$$

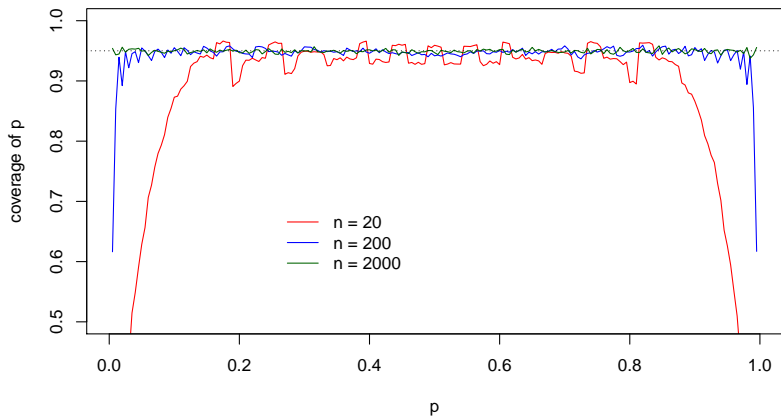
**Exercise:** Let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Bernoulli}(p)$  and set  $\hat{p}_n = n^{-1}(X_1 + \dots + X_n)$ .

- 1 Letting  $g(z) = z/(1 - z)$ , find the asymptotic dist. of  $\sqrt{n}(g(\hat{p}_n) - g(p))$ .
- 2 Propose a CI for the odds  $p/(1 - p)$  and check its performance in simulation.
- 3 Do the same for the log-odds  $\log(p/(1 - p))$ .
- 4 Study the asymptotic behavior of  $\sin^{-1}(\sqrt{\hat{p}_n})$  (*variance stabilizing*).
- 5 Check performance of CI for  $p$  based on variance-stabilization.









**Exercise:** Let  $X_1, \dots, X_n$  be a rs from a dist. with mean  $\mu$  and variance  $\sigma^2 < \infty$ .

- 1 Describe the behavior of  $\sqrt{n}(\bar{X}_n^{-1} - \mu^{-1})$  as  $n \rightarrow \infty$ .
- 2 Give a seq. of rvs  $\{W_n\}_{n \geq 1}$  such that  $\sqrt{n}(\bar{X}_n^{-1} - \mu^{-1})/W_n \xrightarrow{D} \text{Normal}(0, 1)$ .
- 3 Propose a large-sample  $(1 - \alpha) \times 100\%$  CI for  $\mu^{-1}$ .

## Theorem (Second-order delta method)

Let  $Y_n$  be a sequence of rvs and  $\theta$  be a real number such that

$$\sqrt{n}(Y_n - \theta) \xrightarrow{D} \text{Normal}(0, \sigma^2) \quad \text{as } n \rightarrow \infty.$$

Then for any fn.  $g$  st  $g'(\theta) = 0$  and  $g''(\theta)$  exists and is not equal to zero, we have

$$n[g(Y_n) - g(\theta)] \xrightarrow{D} \frac{g''(\theta)\sigma^2}{2} \cdot W, \quad W \sim \chi_1^2 \quad \text{as } n \rightarrow \infty.$$

**Exercise:** Prove using Taylor's theorem, which, with  $g'(\theta) = 0$ , gives

$$g(y) = g(\theta) + \frac{1}{2}g''(\theta)(y - \theta)^2 + R_2(y, \theta), \quad \text{with } \lim_{y \rightarrow \theta} \frac{R_2(y, \theta)}{(y - \theta)^2} = 0.$$

**Exercise:** Let  $X_1, \dots, X_n$  be a rs from a dist. with mean  $\mu$  and variance  $\sigma^2 < \infty$ . Consider the large-sample behavior of  $\bar{X}_n^2 - \mu^2$  under  $\mu = 0$  and  $\mu \neq 0$ .

## Theorem (Multivariate central limit theorem)

For  $n \geq 1$ , let  $\mathbf{X}_1, \dots, \mathbf{X}_n$  be ind. realizations of the rvec  $\mathbf{X} = (X_1, \dots, X_d)^T$  with  $\mathbb{E}\mathbf{X} = \boldsymbol{\mu}$  and  $\text{Cov}\mathbf{X} = \boldsymbol{\Sigma} < \infty$ . Set  $\bar{\mathbf{X}}_n = n^{-1} \sum_{i=1}^n \mathbf{X}_i$ . Then

$$\sqrt{n}(\bar{\mathbf{X}}_n - \boldsymbol{\mu}) \xrightarrow{D} \text{Normal}(0, \boldsymbol{\Sigma}) \quad \text{as } n \rightarrow \infty.$$

By  $\boldsymbol{\Sigma} < \infty$  we mean the largest eigenvalue of  $\boldsymbol{\Sigma}$  is finite.

**Example:** Let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be independent copies of  $(X, Y)$ , where

$$\mathbb{E}X = \mu_X, \quad \mathbb{E}Y = \mu_Y, \quad \text{Var}X = \sigma_X^2, \quad \text{Var}Y = \sigma_Y^2, \quad \text{Cov}(X, Y) = \sigma_{XY}.$$

Give the asymptotic distribution of  $\sqrt{n} \left( \begin{bmatrix} \bar{X}_n \\ \bar{Y}_n \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \right)$  as  $n \rightarrow \infty$ .

## Theorem (Multivariate delta method)

Let  $\mathbf{Y}_n = (Y_{n1}, \dots, Y_{nd})^T$ ,  $n \geq 1$ , be a sequence of rvecs and  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)^T$  be a vector of real numbers such that

$$\sqrt{n}(\mathbf{Y}_n - \boldsymbol{\theta}) \xrightarrow{D} \text{Normal}(\mathbf{0}, \boldsymbol{\Sigma}) \quad \text{as } n \rightarrow \infty.$$

Then for any function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\dot{\mathbf{g}}(\boldsymbol{\theta}) = \left[ \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_1}, \dots, \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_d} \right]^T,$$

exists and is not equal to zero, we have

$$\sqrt{n}[g(\mathbf{Y}_n) - g(\boldsymbol{\theta})] \xrightarrow{D} \text{Normal}(0, [\dot{\mathbf{g}}(\boldsymbol{\theta})]^T \boldsymbol{\Sigma} [\dot{\mathbf{g}}(\boldsymbol{\theta})]) \quad \text{as } n \rightarrow \infty.$$

**Exercise:** Let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be independent copies of  $(X, Y)$ , where

$$\mathbb{E}X = \mu_X, \quad \mathbb{E}Y = \mu_Y, \quad \text{Var} X = \sigma_X^2, \quad \text{Var} Y = \sigma_Y^2, \quad \text{Cov}(X, Y) = \sigma_{XY}.$$

Give the asymptotic distribution of  $\sqrt{n}(\bar{X}_n/\bar{Y}_n - \mu_X/\mu_Y)$  as  $n \rightarrow \infty$ .

**Exercise:** Let  $X \sim \text{Binomial}(n, p_1)$  and  $Y \sim \text{Binomial}(n, p_2)$ , with  $X \perp\!\!\!\perp Y$ .

- 1 Let  $\hat{p}_1 = X/n$  and  $\hat{p}_2 = Y/n$  and show that

$$\sqrt{n} \begin{bmatrix} \hat{p}_1/(1 - \hat{p}_1) - p_1/(1 - p_1) \\ \hat{p}_2/(1 - \hat{p}_2) - p_2/(1 - p_2) \end{bmatrix} \xrightarrow{D} \text{Normal}(0, \vartheta) \quad \text{as } n \rightarrow \infty,$$

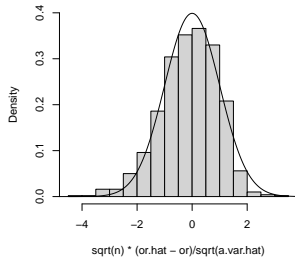
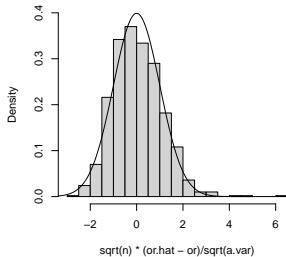
where

$$\vartheta = \frac{p_1/(1 - p_1)}{p_2/(1 - p_2)} \left[ \left( \frac{1 - p_2}{p_2} \right) \frac{1}{(1 - p_1)^2} + \left( \frac{p_1}{1 - p_1} \right) \frac{1}{p_2^2} \right].$$

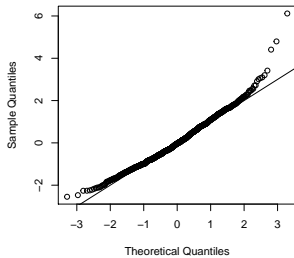
- 2 Find the asymptotic distribution of

$$\sqrt{n} \left[ \log \left( \frac{\hat{p}_1/(1 - \hat{p}_1)}{\hat{p}_2/(1 - \hat{p}_2)} \right) - \log \left( \frac{p_1/(1 - p_1)}{p_2/(1 - p_2)} \right) \right]$$





Normal Q-Q Plot



Normal Q-Q Plot

