STAT 712 fa 2022 Lec 15 slides

Central limit theorem, Slutzky's theorem, delta method

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Recall:

If $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0,1), \qquad \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}, \qquad \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$$

What if $X_1, \dots, X_n$ are *not* sampled from a Normal distribution?

Central limit theorem informally stated

If $X_1, \dots, X_n$ are iid with mean μ and variance $\sigma^2 < \infty$, then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$ behaves more and more like $Z \sim \mathrm{N}(0,1)$ for larger and larger n.

Same is true of $\dfrac{\bar{X}_n - \mu}{\hat{\sigma}_n/\sqrt{n}}$, where $\boxed{\hat{\sigma}_n \xrightarrow{\text{p}} \sigma.}$

Can use to build large-n CI for μ when population non-Normal.

Convergence in distribution

Seq. of rvs $Y_1, Y_2, \ldots$ w/cdfs $F_{Y_1}, F_{Y_2}, \ldots$ *converges in distribution* to $Y \sim F_Y$ if

$$\lim_{n \to \infty} F_{Y_n}(y) = \boxed{F_Y(y),}$$

for all $y \in \mathbb{R}$ at which $F_Y(y)$ is continuous. We write $Y_n \xrightarrow{\text{D}} Y$.

Formalizes "behaves more and more like".

Refer to dist. with cdf F_Y as the *asymptotic distribution* or *limiting distribution*.

Theorem (Convergence in prob. implies convergence in dist.)

For any $\{X_n\}_{n \geq 1}$ and X we have $X_n \xrightarrow{p} X \implies X_n \xrightarrow{D} X$. (See ex. 5.40 of CB)

Exercise: For $n \geq 1$, let $X_1, \dots, X_n \overset{\text{ind}}{\sim}$ Exponential(λ) and let

$$\boxed{Y_n} = (X_{(n)} - \lambda \log n)/\lambda \quad \text{and} \quad Y \sim F_Y(y) = \underline{e^{-e^{-y}}} \text{ for } y \in \mathbb{R}.$$

"Gumbel"

Show that $Y_n \xrightarrow{\text{D}} Y$.

Gumby.

Y_n is an *asymptotic pivot quantity*: limiting dist. known and parameter-free.

$X_1, \dots, X_n \sim \text{Exponential}(\lambda)$, $\quad Y_n = \dfrac{X_{(n)} - \lambda \log n}{\lambda}$

show $\quad \lim_{n \to \infty} F_{Y_n}(y) = F_Y(y) = e^{-e^{-y}}$

$Y = \#\, X_1, \dots, X_n \le x$

$\Rightarrow Y \sim \text{Binomial}(n, F_X(x))$

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(\text{all } n \text{ of } X_1, \dots, X_n \le x)$$

$$= \left[F_X(x)\right]^n$$

$$p_Y(y) = \binom{n}{y} (F_X(x))^y (1 - F_X(x))^{n-y}$$

$$= \left[1 - e^{-x/\lambda}\right]^n$$

$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \mathbb{1}(x > 0)$, $\quad F_X(x) = \int_0^x \frac{1}{\lambda} e^{-t/\lambda}\, dt = \underline{1 - e^{-x/\lambda}}$ for $x > 0$.

$$F_{Y_n}(y) = P(Y_n \le y)$$

$$= P\left(\frac{X_{(n)} - \lambda \log n}{\lambda} \le y\right)$$

$$= P(X_{(n)} \le \lambda y + \lambda \log n)$$

$$= P(X_{(n)} \le \lambda(y + \log n))$$

$$= F_{X_{(n)}}(\lambda(y + \log n))$$

$$= \left[1 - e^{-\frac{\lambda(y+\log n)}{\lambda}}\right]^n$$

$$= \left[1 - e^{-y} e^{-\log n}\right]^n$$

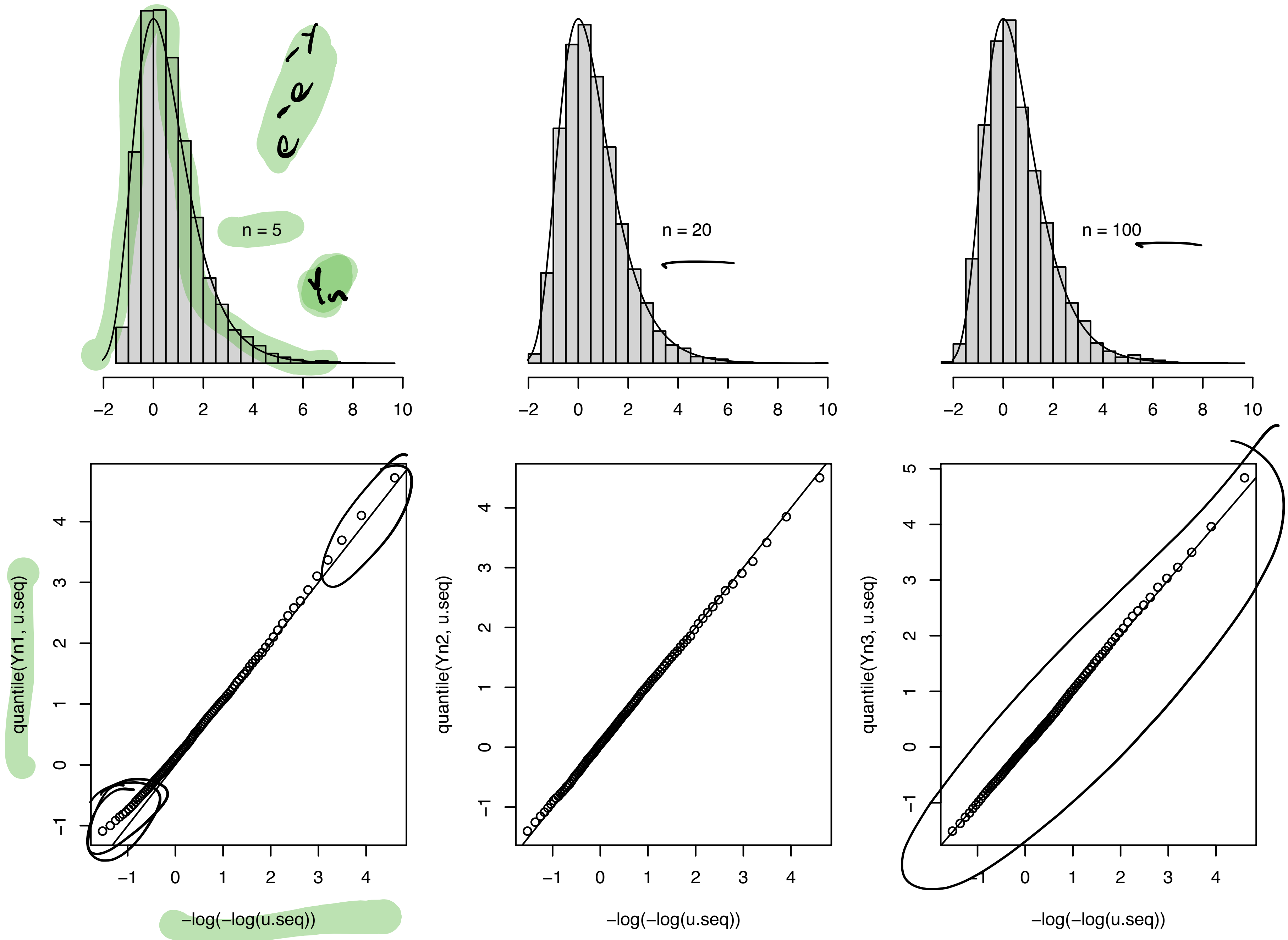
$$= \left[1 - \frac{e^{-y}}{e^{\log n}}\right]^n$$

$$\left(1 + \frac{a}{n}\right)^n \to e^a$$

as $n \to \infty$.

$$= \left[1 - \frac{e^{-y}}{n}\right]^n$$

$$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty}\left[1 - \frac{e^{-y}}{n}\right]^n = e^{-e^{-y}}$$

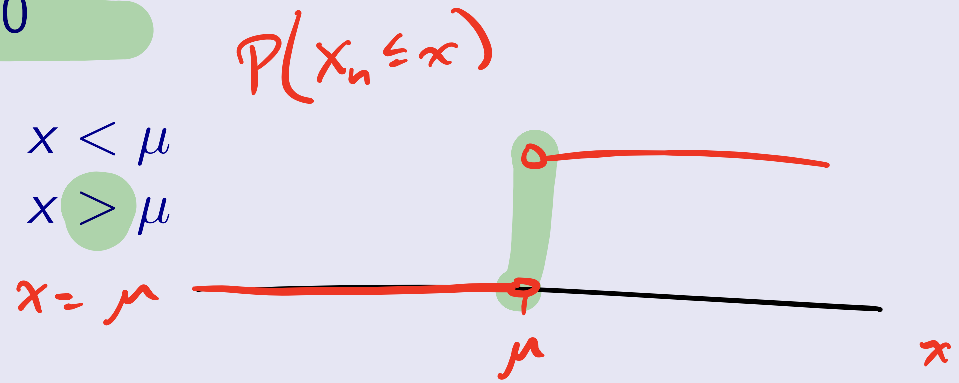

$e^{-e^{-y}}$
n = 5
Y_n
n = 20
n = 100
–2 0 2 4 6 8 10
quantile(Yn1, u.seq)
quantile(Yn2, u.seq)
quantile(Yn3, u.seq)
–log(–log(u.seq))

Theorem (Convergence in distribution to a constant)

For a sequence of rvs $\{X_n\}_{n\geq 1}$, the following statements are equivalent

1. $\lim_{n\to\infty} P(|X_n - \mu| > \varepsilon) = 0$
2. $\lim_{n\to\infty} \boxed{P(X_n \leq x)} = \begin{cases} 0 & x < \mu \\ 1 & x > \mu \end{cases}$

(See ex. 5.41 in CB)



So $X_n \xrightarrow{p} \mu \iff X_n \xrightarrow{D} X$, where X is a rv equal to μ with probability 1.

Discuss: The second statement has $x > \mu$ instead of $x \geq \mu$. Important?

A distribution which puts all its mass on a single value is a *degenerate* distribution.

Theorem (Central limit theorem)

If $X_1, \dots, X_n$ a rs from a dist. with mgf defined in a neighborhood of zero, then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} Z \sim \text{Normal}(0,1) \text{ as } n \to \infty,$$

where μ and σ are the population mean and standard deviation.

$$M_X(t) = \mathbb{E}\, e^{tX} = \mathbb{E}\left[1 + tX + \frac{t^2X^2}{2} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \cdots\right]$$

Mgf assumption not needed. Can prove CLT requiring only $\sigma^2 < \infty$ (advanced).

Taylor expand e^z around $z = 0$.

Exercise: Prove using mgfs.

$$\lim_{n\to\infty} M_{\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}}(t) \longrightarrow M_Z(t) = e^{t^2/2}$$

$$\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} = \sqrt{n}\left(\frac{\bar{X}_n-\mu}{\sigma}\right)$$

$$= \sqrt{n}\left(\frac{\frac{1}{n}\sum X_i - \mu}{\sigma}\right)$$

$$= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)\right)$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i \,, \qquad Y_i = \frac{X_i-\mu}{\sigma}.$$

So want to show

$$M_{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i}(t) \longrightarrow e^{t^2/2} \quad \text{as} \quad n\to\infty.$$

Write

$$M_{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i}(t) = M_{\sum_{i=1}^{n} Y_i}(t/\sqrt{n})$$

$$= \left[M_{Y_1}(t/\sqrt{n})\right]^n$$

$$= \left[\mathbb{E}\, e^{Y_1 t/\sqrt{n}} \right]^n \qquad Y_i = \frac{X_i - \mu}{\sigma}$$

$$\mathbb{E}\, Y_1^2 = \mathrm{Var}\, Y_1 = \mathrm{Var}\left(\frac{X_1 - \mu}{\sigma}\right) = 1$$

$$= \left[\mathbb{E}\left(1 + Y_1 \frac{t}{\sqrt{n}} + \frac{Y_1^2 (t/\sqrt{n})^2}{2!} + \frac{Y_1^3 (t/\sqrt{n})^3}{3!} + \frac{Y_1^4 (t/\sqrt{n})^4}{4!} + \dots \right) \right]^n$$

$$= \left[1 + 0 + \frac{t^2/2}{n} + \frac{\mathbb{E}\, Y_1^3\, t^3}{3!\,(\sqrt{n})^3} + \frac{\mathbb{E}\, Y_1^4\, t^4}{4!\, n^2} + \dots \right]^n$$

$$= \left[1 + \frac{t^2/2 + \frac{\mathbb{E}\, Y_1^3\, t^3}{3!\sqrt{n}} + \frac{\mathbb{E}\, Y_1^4\, t^4}{4!\, n} + \dots}{n} \right]^n$$

$$\to e^{t^2/2} \quad \text{as } n \to \infty. \qquad \left[1 + \frac{t^2/2}{n} \right]^n \to e^{t^2/2}$$

Result:

If $a_n \to a$ as $n \to \infty$

Then

$\left[1 + \frac{a_n}{n} \right]^n \to e^a$

as $n \to \infty$

$$X = (b-a)Z + a \sim \text{Unif}(a,b)$$

$$\mu = \frac{a+b}{2}$$

$$\sigma^2 = \text{Var}\, X = \text{Var}\left((b-a)Z\right) = (b-a)^2 \,\text{Var}\, Z = \frac{(b-a)^2}{12}$$

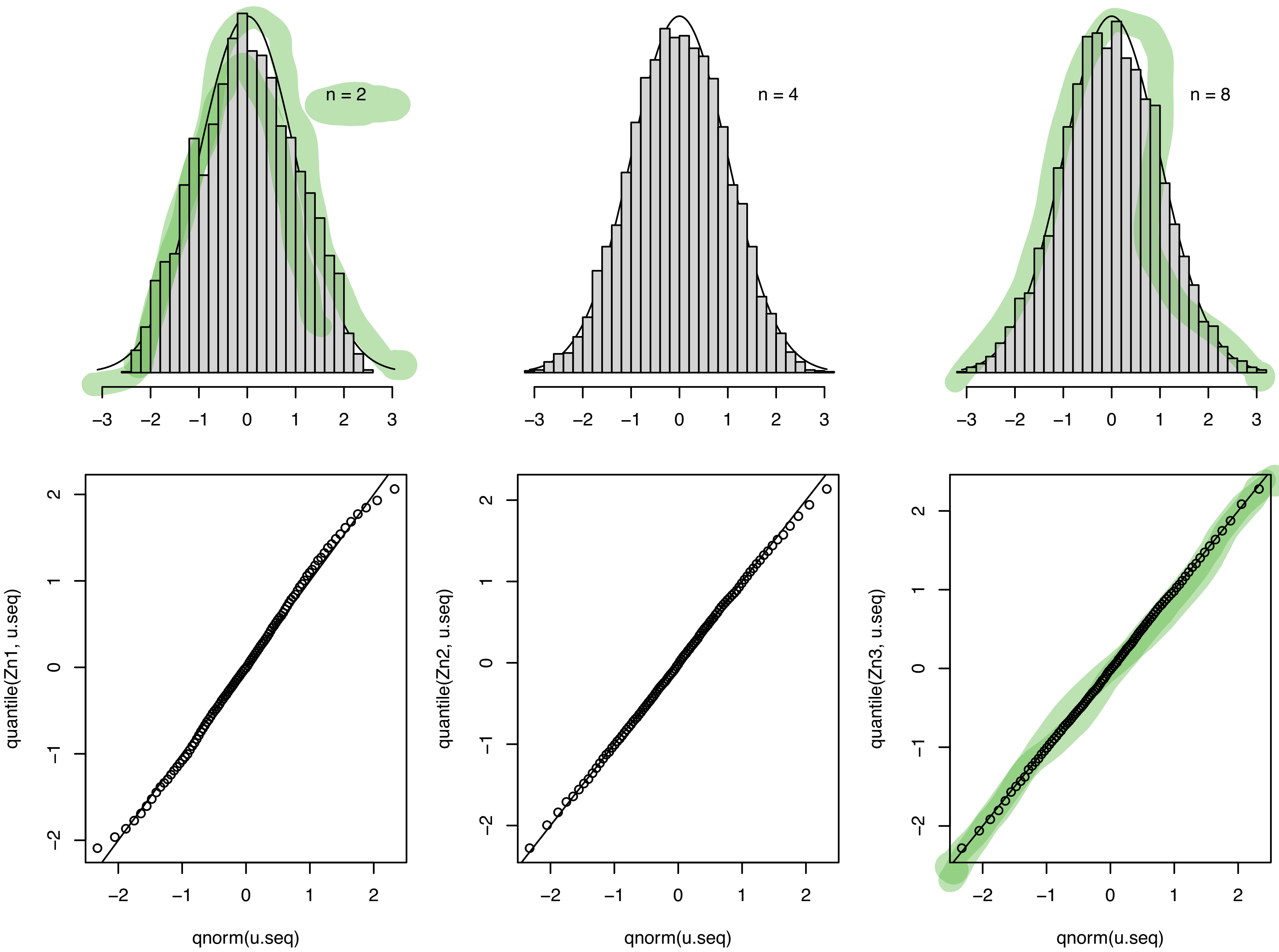
$$Z = \frac{X-a}{b-a} \sim U(0,1)$$

Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(a, b)$.

1. Give a function of $\bar{X}_n$ which is asymptotically standard Normal.
2. Run some simulations to investigate the rate of convergence.

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \sqrt{n}\left(\frac{\bar{X}_n - \frac{a+b}{2}}{\sqrt{\frac{(b-a)^2}{12}}}\right) \xrightarrow{D} N(0,1)$$

as $n \to \infty$.


n = 2
n = 4
n = 8
−3 −2 −1 0 1 2 3
quantile(Zn1, u.seq)
quantile(Zn2, u.seq)
quantile(Zn3, u.seq)
qnorm(u.seq)
−2 −1 0 1 2

Theorem (Slutzky's theorem)

If $X_n \xrightarrow{D} X$ *and* $Y_n \xrightarrow{p} a$, *then*

1. $X_n + Y_n \xrightarrow{D} X + a$
2. $X_n Y_n \xrightarrow{D} Xa$
3. $X_n / Y_n \xrightarrow{D} X/a$ *provided* $a \neq 0$

Estimate with $\bar{x}_n$?

Exercise: For $n \geq 1$, let $X_1, \dots, X_n \overset{\text{ind}}{\sim}$ Exponential(λ) and let

$$Y_n = (X_{(n)} - \lambda \log n)/\lambda \quad \text{and} \quad Y \sim F_Y(y) = e^{-e^{-y}} \text{ for } y \in \mathbb{R}.$$

Consider whether $Y_n = (X_{(n)} - \bar{X}_n \log n)/\bar{X}_n$ converges in distribution to Y.

$$\tilde{Y}_n = \left(X_{(n)} - \bar{X}_n \log n\right) / \bar{X}_n$$

$$= \frac{\left(X_{(n)} - \lambda \log n + \lambda \log n - \bar{X}_n \log n\right)}{\lambda} \cdot \frac{\lambda}{\bar{X}_n}$$

$\bar{X}_n \xrightarrow{P} \lambda \Rightarrow \frac{\lambda}{\bar{X}_n} \xrightarrow{P} 1$

$$= \underbrace{\underbrace{\left(\frac{X_{(n)} - \lambda \log n}{\lambda}\right)}_{Y_n \xrightarrow{D} Y \sim F_Y(y) = e^{-e^{-y}}} \underbrace{\left(\frac{\lambda}{\bar{X}_n}\right)}_{\xrightarrow{P} 1}}_{\xrightarrow{D} Y} + \underbrace{\underbrace{\frac{(\lambda - \bar{X}_n) \log n}{\lambda}}_{\xrightarrow{P} 0} \cdot \underbrace{\frac{\lambda}{\bar{X}_n}}_{\xrightarrow{P} 1}}_{\xrightarrow{P} 0}$$

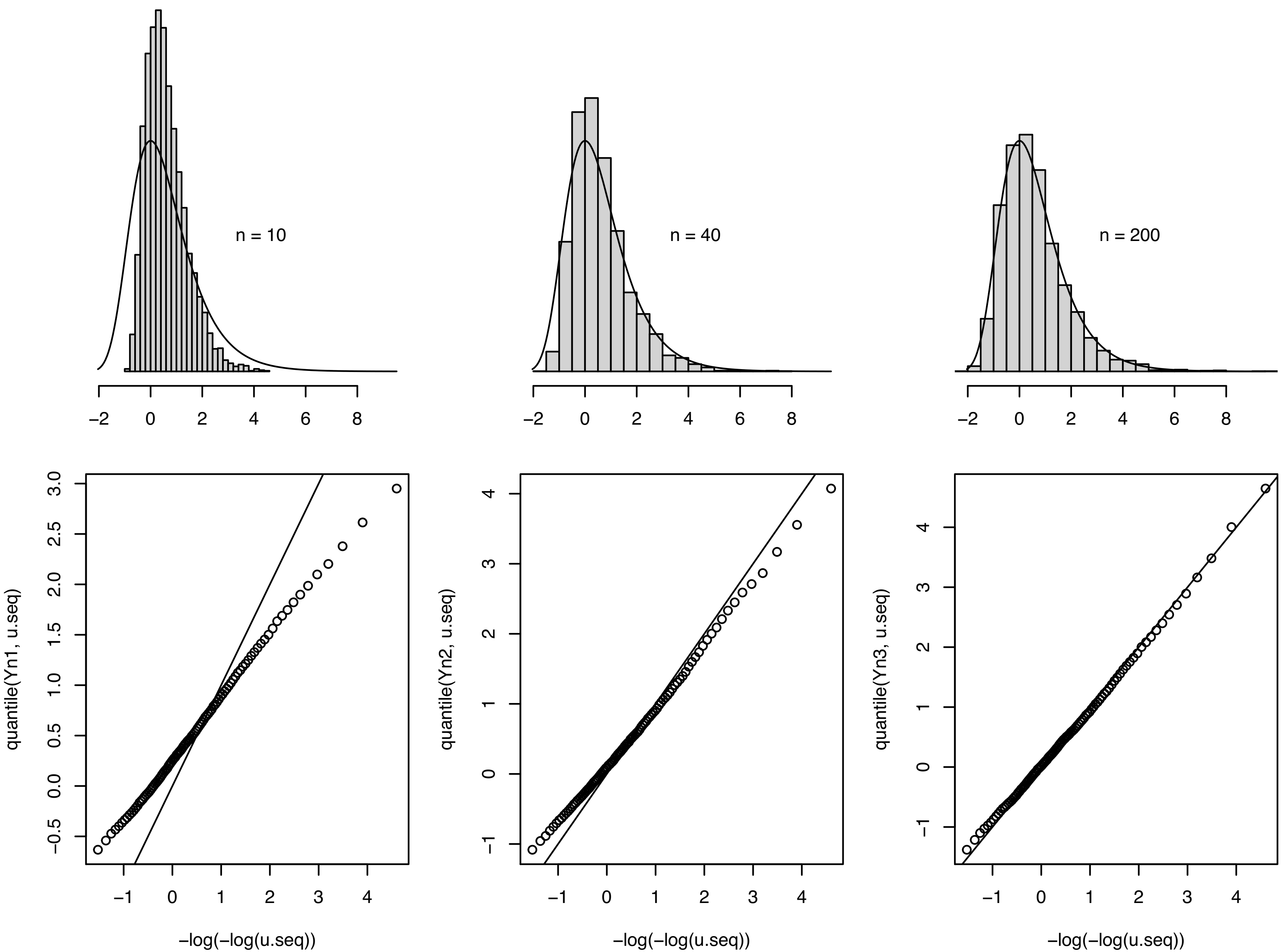
$$\xrightarrow{D} Y \sim F_Y(y) = e^{-e^{-y}}$$

$$\mathbb{E}\left[(\lambda - \bar{X}_n) \log n\right] = 0$$

$$\mathrm{Var}\left[(\lambda - \bar{X}_n) \log n\right] = (\log n)^2 \, \mathrm{Var}\, \bar{X}_n = (\log n)^2 \frac{\mathrm{Var}\, X_1}{n}$$

$$\longrightarrow 0$$

as $n \to \infty$.


n = 10
n = 40
n = 200
–2 0 2 4 6 8
quantile(Yn1, u.seq)
quantile(Yn2, u.seq)
quantile(Yn3, u.seq)
–log(–log(u.seq))
–log(–log(u.seq))
–log(–log(u.seq))

Theorem (Corollary of Slutzky's theorem and central limit theorem)

Let $X_1, \dots, X_n$ be a rs from a dist. with mean μ and variance $\sigma^2 < \infty$. Then

$$\boxed{\frac{\bar{X}_n - \mu}{\hat{\sigma}_n/\sqrt{n}}} \xrightarrow{D} Z \sim \text{Normal}(0,1) \text{ as } n \to \infty,$$

provided $\hat{\sigma}_n \xrightarrow{p} \sigma$.

Exercise:

1. Show how the CLT and Slutzky's theorem imply the above.
2. Illustrate this result in simulation for $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(a, b)$.

$$\frac{\bar{X}_n - \left(\frac{a+b}{2}\right)}{S_n/\sqrt{n}} \xrightarrow{D} N(0,1)$$

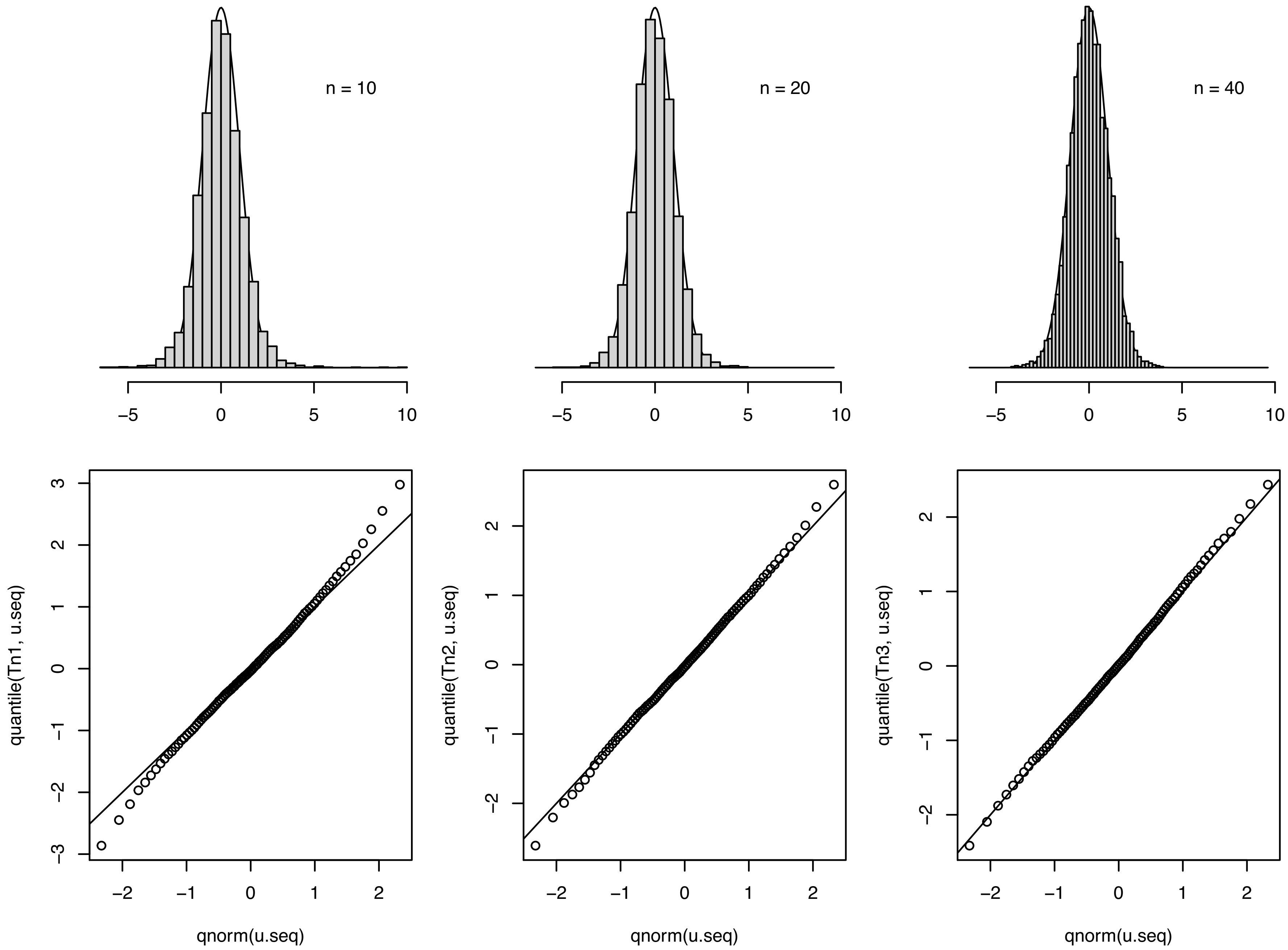
1

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1) \qquad \text{Central lim. Thm.}$$

Assume $\hat{\sigma}_n \xrightarrow{p} \sigma$.

Then

$$\frac{\bar{X}_n - \mu}{\hat{\sigma}_n/\sqrt{n}} = \underbrace{\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}_{\xrightarrow{D} Z \sim N(0,1)} \underbrace{\left(\frac{\sigma}{\hat{\sigma}_n}\right)}_{\xrightarrow{p} 1} \xrightarrow{D} Z \sim N(0,1).$$



n = 10
n = 20
n = 40
−5
0
5
10
quantile(Tn1, u.seq)
quantile(Tn2, u.seq)
quantile(Tn3, u.seq)
qnorm(u.seq)
−3
−2
−1
0
1
2
3

$\text{Var}\, S_n^2 = \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\theta_2^2\right) \to 0,$

$\theta_4 = \mathbb{E}(X - \mathbb{E}X)^4 < \infty$

$\mathbb{E}\, S_n^2 = \sigma^2$

$\theta_2 = \text{Var}\, X$

Exercises: Derive $\boxed{\text{asymptotic confidence}}$ intervals for

1. μ based on $X_1, \ldots, X_n$ iid with mean μ and finite 4th moment.
2. p based on $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$.

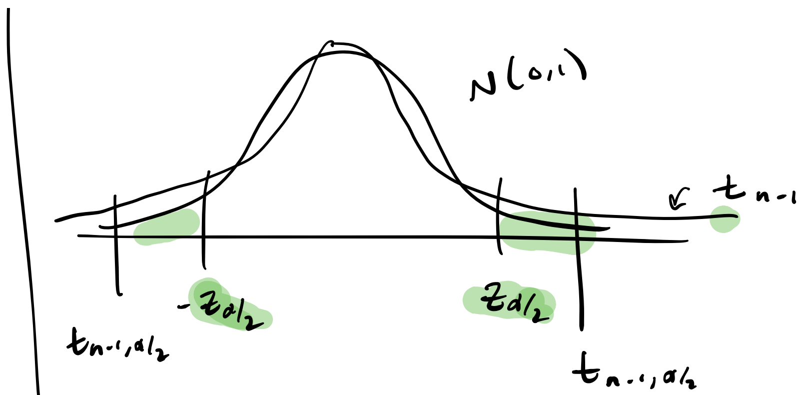
$\sim t_{n-1}$ if $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.

$S_n \xrightarrow{p} \sigma$ $\iff$

(1) $\dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \xrightarrow{D} N(0,1)$, provided $\boxed{S_n^2 \xrightarrow{p} \sigma^2}$.

Require 4th moment finite. i.e. $\mathbb{E}X^4 < \infty$.

$P\left(-z_{\alpha/2} < \dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}} < z_{\alpha/2}\right) \to 1-\alpha$ as $n \to \infty$.



$$P\left(\bar{X}_n - z_{\alpha/2}\frac{S_n}{\sqrt{n}} < \mu < \bar{X}_n + z_{\alpha/2}\frac{S_n}{\sqrt{n}}\right) \to 1-\alpha$$

as $n \to \infty$.

So $\bar{X}_n \pm z_{\alpha/2}\frac{S_n}{\sqrt{n}}$ is a $(1-\alpha)\cdot 100\%$ C.I. for μ

② $X_1, \dots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$. $\sigma^2 = p(1-p)$

$$\frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{D} N(0,1) \quad \text{as } n \to \infty, \text{ by C.L.T.}$$

$\hat{p}_n = \bar{X}_n$

$$\hat{p}_n(1-\hat{p}_n) \xrightarrow{P} p(1-p) \quad \text{as } n \to \infty.$$

$\hat{p}_n \xrightarrow{P} p$ by WLLN

$g(x) = x(1-x)$ is cont.

$g(\hat{p}_n) \xrightarrow{P} g(p)$

Slutzky's theorem gives

$$\frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \xrightarrow{D} N(0,1).$$

So

$$P\left(z_{\alpha/2} < \frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} < z_{\alpha/2} \right) \longrightarrow 1-\alpha$$

as $n \to \infty$.

$$P\left(\hat{p}_n - z_{\alpha/2}\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}} < p < \hat{p}_n + z_{\alpha/2}\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}} \right) \longrightarrow 1-\alpha$$

as $n \to \infty$.

So, a asymptotic C.I. for p is

$$\hat{p}_n \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}$$

Exercise: For $n = 10, 20, 40, 80, 160, 320$, simulate coverage of

$$\bar{X}_n \pm z_{0.025} \cdot S_n/\sqrt{n}$$

for mean of Gamma($\alpha = 2, \beta = 4$) distribution.

$EX = \alpha\beta = 2 \cdot 4 = 8$

n	10	20	40	80	160	320
coverage	0.887	0.925	0.934	0.938	0.947	0.947

The *coverage* of a CI is the probability with which it contains its target.

1. Convergence in distribution
2. Central limit theorem, Slutzky's theorem
3. Delta method

Theorem (Delta method)

Let Y_n *be a sequence of rvs and* θ *be a real number such that*

$$\boxed{\sqrt{n}(Y_n - \theta)} \xrightarrow{D} \text{Normal}(0, \sigma^2) \quad \text{as } n \to \infty.$$

Then for any function g *such that* $g'(\theta)$ *exists and is not equal to zero, we have*

$$\boxed{\sqrt{n}[g(Y_n) - g(\theta)]} \xrightarrow{D} \text{Normal}(0, [g'(\theta)]^2\sigma^2) \quad \text{as } n \to \infty.$$

Exercise: Prove using Taylor's theorem, which gives

$$g(y) = g(\theta) + g'(\theta)(y - \theta) + R(y, \theta), \quad \text{with } \lim_{y \to \theta} \frac{R(y, \theta)}{(y - \theta)} = 0.$$

$$g(Y_n) = g(\theta) + g'(\theta)(Y_n - \theta) + R(Y_n, \theta)$$

Now write

$$\sqrt{n}\, g(Y_n) = \sqrt{n}\, g(\theta) + g'(\theta)\sqrt{n}(Y_n - \theta) + \sqrt{n}\, R(Y_n, \theta).$$

$\Leftrightarrow$

$$\underbrace{\sqrt{n}\left(g(Y_n) - g(\theta)\right)} = \boxed{g'(\theta)\, \underbrace{\sqrt{n}(Y_n - \theta)}_{\xrightarrow{P} N(0, \sigma^2)}} + \cancel{\sqrt{n}\, R(Y_n, \theta)}$$

$\xrightarrow{P} N\left(0, [g'(\theta)]^2 \sigma^2\right)$

$$\sqrt{n}\, R(Y_n, \theta) = \underbrace{\boxed{\sqrt{n}(Y_n - \theta)}_{\xrightarrow{P} N(0, \sigma^2)} \; \boxed{\frac{R(Y_n, \theta)}{(Y_n - \theta)}}_{\xrightarrow{P} 0 \text{ since } Y_n \xrightarrow{P} \theta.}}_{\xrightarrow{P} 0 \text{ by Slutzky's thm.}}$$

First, we have, from C.L.T.

$$\sqrt{n}(\hat{p}_n - p) \xrightarrow{D} N(0, p(1-p))$$

"Odds" are $\frac{p}{1-p}$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$ and set $\hat{p}_n = n^{-1}(X_1 + \dots + X_n)$.

1. Letting $g(z) = z/(1-z)$, find the asymptotic dist. of $\sqrt{n}(g(\hat{p}_n) - g(p))$.
2. Propose a CI for the odds $p/(1-p)$ and check its performance in simulation.
3. Do the same for the log-odds $\log(p/(1-p))$. $g(z) = \log\left(\frac{z}{1-z}\right)$
4. Study the asymptotic behavior of $\sin^{-1}(\sqrt{\hat{p}_n})$ (*variance stabilizing*).
5. Check performance of CI for p based on variance-stabilization.

1 $$\sqrt{n}\left(\frac{\hat{p}_n}{1-\hat{p}_n} - \frac{p}{1-p}\right) \xrightarrow{D} \text{Normal}\left(0, [g'(p)]^2 \sigma^2\right)$$

$$g(z) = \frac{z}{1-z} \quad g'(z) = \frac{1}{1-z} + \frac{(-1)z(-1)}{(1-z)^2} = \frac{1}{1-z} + \frac{z}{(1-z)^2} = \frac{1-z+z}{(1-z)^2} = \frac{1}{(1-z)^2}$$

$$[g'(p)]^2\sigma^2 = \left[\frac{1}{(1-p)^2}\right]^2 p(1-p) = \frac{p}{(1-p)^3}$$

$$\sqrt{n}\left(\frac{\hat{p}_n}{1-\hat{p}_n} - \frac{p}{1-p}\right) \xrightarrow{D} \text{Normal}\left(0, \frac{p}{(1-p)^3}\right)$$

Then

$$\frac{\sqrt{n}\left(\frac{\hat{p}_n}{1-\hat{p}_n} - \frac{p}{1-p}\right)}{\sqrt{\frac{\hat{p}_n}{(1-\hat{p}_n)^3}}} \xrightarrow{D} N(0,1)$$

by Slutzky's thm, since $\frac{\hat{p}_n}{(1-\hat{p}_n)^3} \xrightarrow{P} \frac{p}{(1-p)^3}$

$$P\left(-z_{\alpha/2} < \frac{\sqrt{n}\left(\frac{\hat{p}_n}{1-\hat{p}_n} - \frac{p}{1-p}\right)}{\sqrt{\frac{\hat{p}_n}{(1-\hat{p}_n)^3}}} < z_{\alpha/2}\right) \to 1-\alpha$$

Gives C.I.

$$\frac{\hat{p}_n}{1-\hat{p}_n} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_n}{(1-\hat{p}_n)^3 n}}.$$

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4 $\sqrt{n}(\hat{p}_n - p) \xrightarrow{D} N(0, p(1-p))$

$$\sqrt{n}\left(\sin^{-1}(\sqrt{\hat{p}_n}) - \sin^{-1}(\sqrt{p})\right) \xrightarrow{D} N\left(0, [g'(p)]^2 p(1-p)\right)$$

$$g(z) = \sin^{-1}(\sqrt{z}) \qquad \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$g'(z) = \frac{1}{\sqrt{1-(\sqrt{z})^2}}\,\frac{1}{2\sqrt{z}} = \frac{1}{\sqrt{1-z}}\,\frac{1}{2\sqrt{z}} = \frac{1}{2\sqrt{z(1-z)}}$$

$$g'(p) = \frac{1}{2\sqrt{p(1-p)}} \qquad [g'(p)]^2 p(1-p) = \left(\frac{1}{2\sqrt{p(1-p)}}\right)^2 p(1-p) = \frac{1}{4}$$

$$\sqrt{n}\left(\sin^{-1}(\sqrt{\hat{p}_n}) - \sin^{-1}(\sqrt{p})\right) \xrightarrow{D} N\left(0, \frac{1}{4}\right)$$

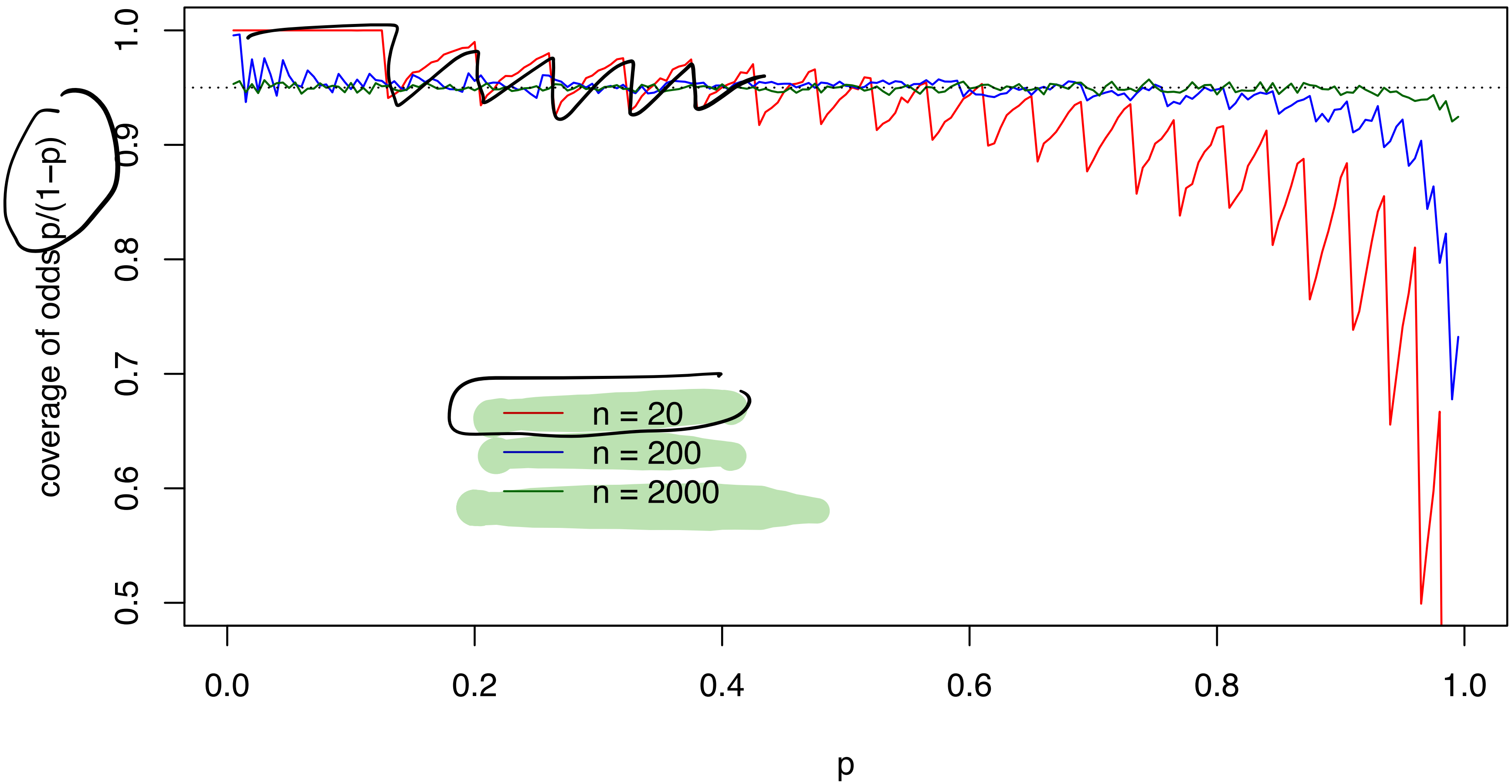
Can we build a C.I. for $\sin^{-1}(\sqrt{p})$?

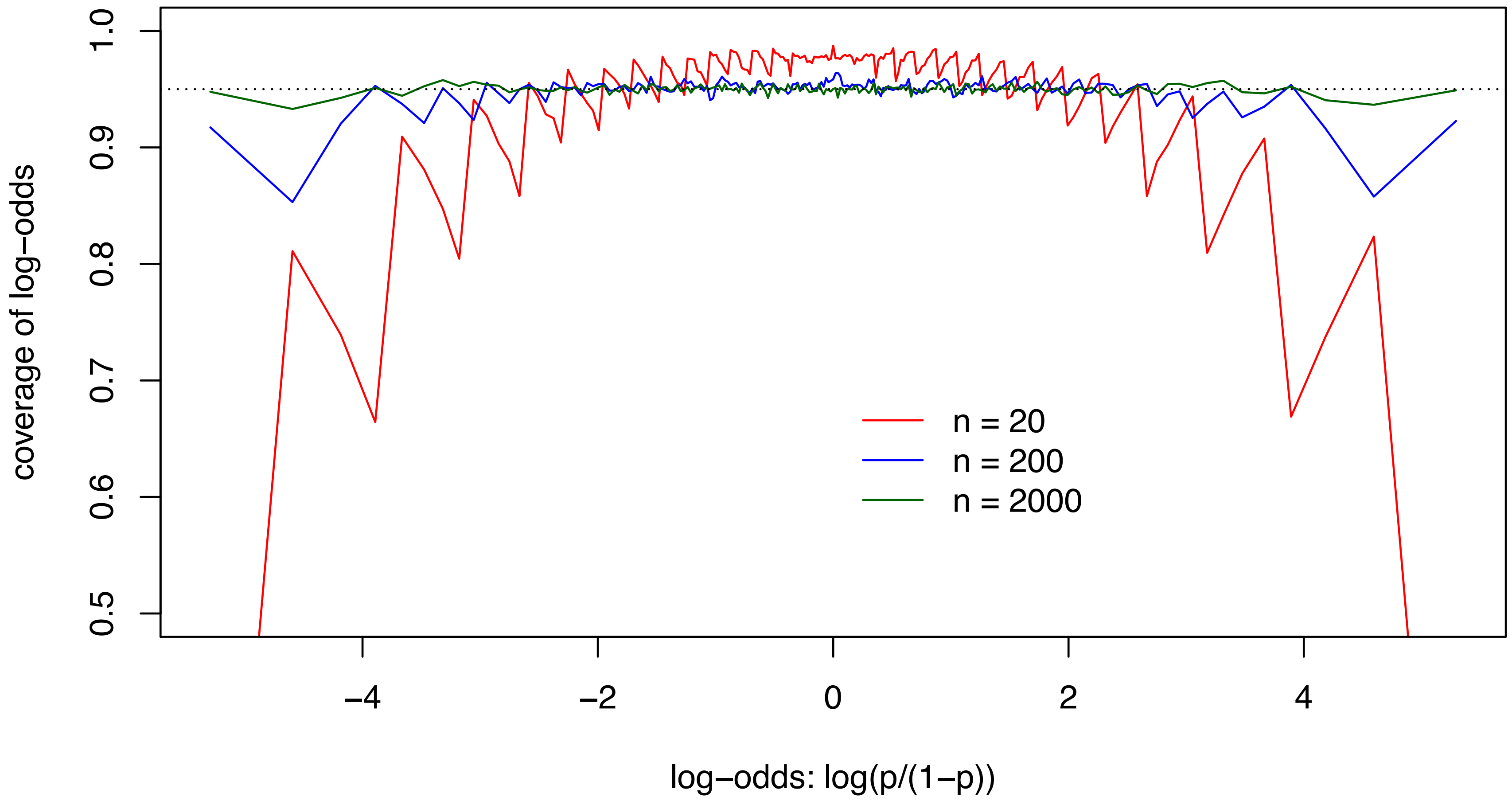
$$P\left(-z_{\alpha/2} < \sqrt{n}\, 2\left(\sin^{-1}(\sqrt{\hat{p}_n}) - \sin^{-1}(\sqrt{p})\right) < z_{\alpha/2}\right) \to 1-\alpha$$

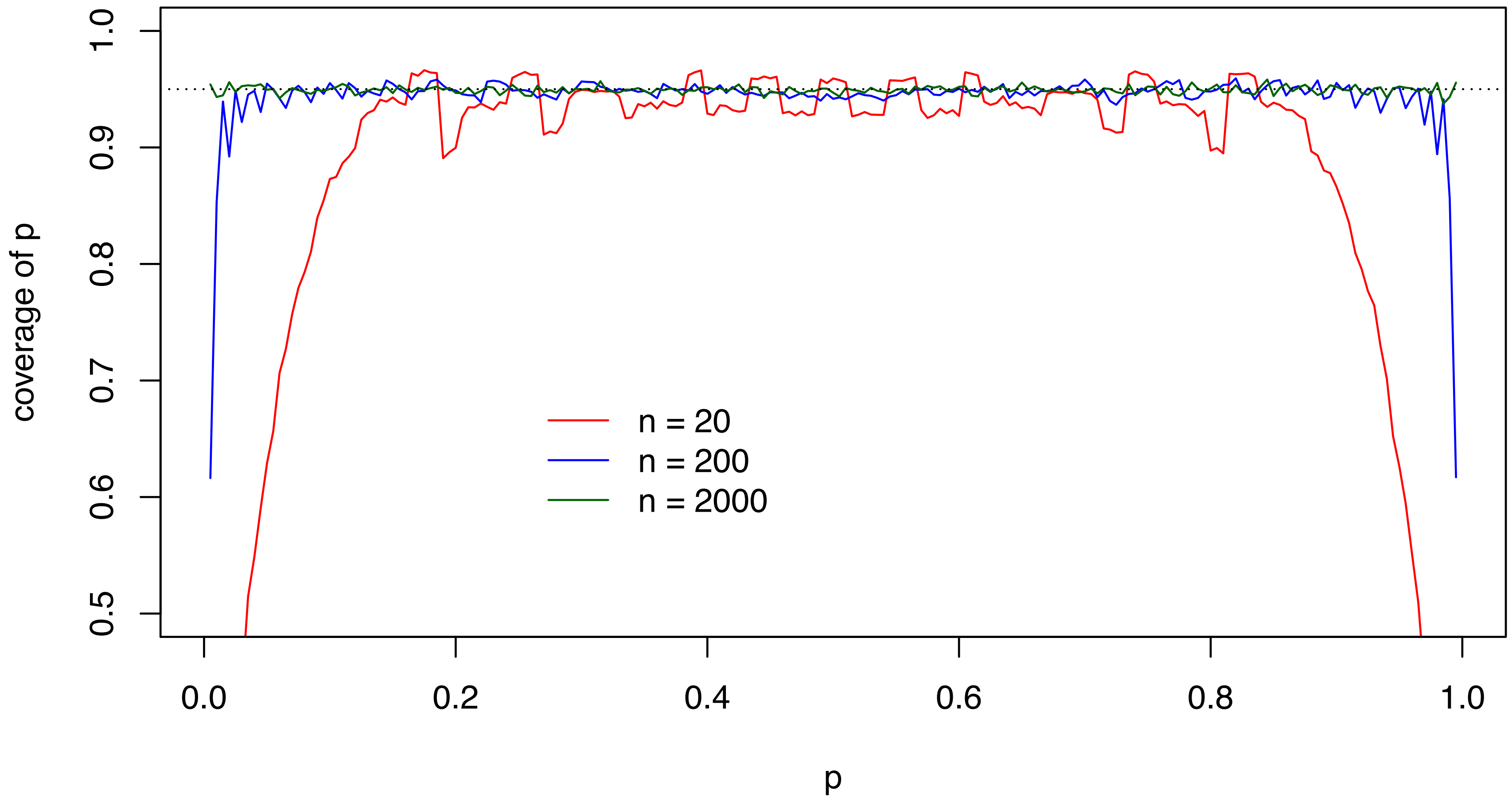
as $n \to \infty$.

$\Rightarrow$

$$\sin^{-1}(\sqrt{\hat{p}_n}) \pm z_{\alpha/2} \frac{1}{2\sqrt{n}}$$ is a $(1-\alpha)100\%$ C.I. for $\sin^{-1}(\sqrt{p})$.


coverage of odds p/(1−p)
1.0
0.9
0.8
0.7
0.6
0.5
0.0
0.2
0.4
0.6
0.8
1.0
p
n = 20
n = 200
n = 2000


1.0
0.9
0.8
0.7
0.6
0.5
coverage of log−odds
−4
−2
0
2
4
log−odds: log(p/(1−p))
n = 20
n = 200
n = 2000


coverage of p
p
0.5
0.6
0.7
0.8
0.9
1.0
0.0
0.2
0.4
0.6
0.8
1.0
n = 20
n = 200
n = 2000

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{D} N(0,1) \quad \text{as } n \to \infty.$$

$$\Longleftrightarrow$$

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2),$$

Exercise: Let $X_1, \ldots, X_n$ be a rs from a dist. with mean μ and variance $\sigma^2 < \infty$.

1. Give a seq. of rvs $\{W_n\}_{n \geq 1}$ such that $\sqrt{n}(\bar{X}_n^{-1} - \mu^{-1})/W_n \xrightarrow{D} \text{Normal}(0, 1)$.
2. Propose a large-sample $(1 - \alpha) \times 100\%$ CI for μ^{-1}.

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\mu}\right) \xrightarrow{D} N\left(0, \underbrace{[g'(\mu)]^2 \sigma^2}_{\left[-\frac{1}{\mu^2}\right]^2 \sigma^2}\right)$$

$$g(x) = \frac{1}{x} \qquad g'(x) = -\frac{1}{x^2}$$

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\mu}\right) \xrightarrow{D} N\left(0, \frac{\sigma^2}{\mu^4}\right)$$

$\sqrt{n}\,(g(Y_n) - g(\theta)) \xrightarrow{D} \text{Normal}(0, [g'(\theta)]^2\sigma^2)$

Theorem (Second-order delta method)

Let Y_n be a sequence of rvs and θ be a real number such that

$$\boxed{\sqrt{n}(Y_n - \theta) \xrightarrow{D} \text{Normal}(0, \sigma^2)} \quad \text{as } n \to \infty.$$

Then for any fn. g st $g'(\theta) = 0$ and $g''(\theta)$ exists and is not equal to zero, we have

$$n[g(Y_n) - g(\theta)] \xrightarrow{D} \boxed{\frac{g''(\theta)\sigma^2}{2}} \cdot W, \quad W \sim \chi^2_1 \quad \text{as } n \to \infty.$$

Exercise: Prove using Taylor's theorem, which, with $g'(\theta) = 0$, gives

$$g(y) = g(\theta) + \frac{1}{2}g''(\theta)(y-\theta)^2 + R_2(y,\theta), \quad \text{with } \lim_{y\to\theta} \frac{R_2(y,\theta)}{(y-\theta)^2} = 0.$$

$\underbrace{g'(\theta)}_{=0}(y-\theta)$

$$g(Y_n) = g(\theta) + \frac{1}{2} g''(\theta)(Y_n - \theta)^2 + R_2(Y_n, \theta)$$

$\Leftrightarrow$

$$\underbrace{n\left(g(Y_n) - g(\theta)\right)} = \frac{\sigma^2 g''(\theta)}{2} \boxed{\frac{n(Y_n - \theta)^2}{\sigma^2}} + \cancel{n R_2(Y_n, \theta)}$$

$$\boxed{\frac{n(Y_n - \theta)^2}{\sigma^2}} = \left[\underbrace{\frac{\sqrt{n}(Y_n - \theta)}{\sigma}}_{\xrightarrow{D} Z \sim N(0,1)}\right]^2 \xrightarrow{D} \chi^2_1$$

$$\xrightarrow{D} \frac{\sigma^2 g''(\theta)}{2} \cdot W$$

$W \sim \chi^2_1$

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2) \quad \text{as} \quad n \to \infty.$$

Exercise: Let $X_1, \ldots, X_n$ be a rs from a dist. with mean μ and variance $\sigma^2 < \infty$. Consider the large-sample behavior of $\bar{X}_n^2 - \mu^2$ under $\mu = 0$ and $\mu \neq 0$.

$g(x) = x^2$ $\qquad$ $g'(x) = 2x$ $\qquad$ $[g'(\mu)]^2 \sigma^2$

For $\mu \neq 0$:

$$\sqrt{n}(\bar{X}_n^2 - \mu^2) \xrightarrow{D} N(0, (2\mu)^2 \sigma^2)$$

For $\mu = 0$: $\qquad$ $g''(x) = 2$

$$n\left(\bar{X}_n^2 - \mu^2\right) \xrightarrow{D} \frac{2 \cdot \sigma^2}{2} \cdot W \quad , \quad W \sim \chi^2_1$$

$$n\left(\bar{X}_n^2 - \mu^2\right) \xrightarrow{D} \sigma^2 \cdot W \quad , \quad W \sim \chi^2_1$$

$$\frac{n\left(\bar{X}_n^2 - \mu^2\right)}{\sigma^2} \xrightarrow{D} \chi^2_1$$

Exercise: Let $X_1, \dots, X_n$ be a rs from a dist. with mean μ and variance $\sigma^2 < \infty$. Consider the large-sample behavior of $\bar{X}_n^2 - \mu^2$ under $\mu = 0$ and $\mu \neq 0$.

$$\operatorname{Cov} \underset{\sim}{X} = \left(\operatorname{Cov}(X_i, X_j) \right)_{1 \le i, j \le d} = \begin{bmatrix} \operatorname{Cov}(X_1,X_1) & \operatorname{Cov}(X_1,X_2) & \cdots & \operatorname{Cov}(X_1,X_d) \\ \operatorname{Cov}(X_2,X_1) & \operatorname{Cov}(X_2,X_2) & & \operatorname{Cov}(X_2,X_d) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Cov}(X_d,X_1) & \operatorname{Cov}(X_d,X_2) & & \operatorname{Cov}(X_d,X_d) \end{bmatrix}$$

$X_1, \ldots, X_n \overset{\text{ind}}{\sim}$ mean μ, var $\sigma^2 < \infty$, then $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$.

Theorem (Multivariate central limit theorem)

For $n \geq 1$, let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be ind. realizations of the rvec $\mathbf{X} = (X_1, \ldots, X_d)^T$ with $\mathbb{E}\mathbf{X} = \boldsymbol{\mu}$ and $\boxed{\text{Cov}\,\mathbf{X}} = \boldsymbol{\Sigma} < \infty$. Set $\bar{\mathbf{X}}_n = n^{-1}\sum_{i=1}^n \mathbf{X}_i$. Then

$\mathbb{E}\begin{bmatrix} X_1 \\ \vdots \\ X_d \end{bmatrix} = \begin{bmatrix} \mathbb{E}X_1 \\ \vdots \\ \mathbb{E}X_d \end{bmatrix} = \underset{\sim}{\mu}$

$$\sqrt{n}(\bar{\mathbf{X}}_n - \boldsymbol{\mu}) \xrightarrow{D} \text{Normal}(\underset{d\times 1}{0}, \underset{d\times d}{\boldsymbol{\Sigma}}) \quad \text{as } n \to \infty.$$

By $\boldsymbol{\Sigma} < \infty$ we mean the largest eigenvalue of $\boldsymbol{\Sigma}$ is finite.

Example: Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent copies of (X, Y), where

$$\mathbb{E}X = \mu_X, \quad \mathbb{E}Y = \mu_Y, \quad \text{Var}\,X = \sigma_X^2, \quad \text{Var}\,Y = \sigma_Y^2, \quad \text{Cov}(X, Y) = \sigma_{XY}.$$

Give the asymptotic distribution of $\sqrt{n}\left(\begin{bmatrix} \bar{X}_n \\ \bar{Y}_n \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}\right)$ as $n \to \infty$.

$\underset{\sim}{X} = \begin{bmatrix} X \\ Y \end{bmatrix}$

$$\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n \quad \text{or} \quad \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_n \\ Y_n \end{bmatrix}$$

$$\Sigma = \operatorname{Cov} \underset{\sim}{X} = \operatorname{Cov}\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} \operatorname{Cov}(X,X) & \operatorname{Cov}(X,Y) \\ \operatorname{Cov}(Y,X) & \operatorname{Cov}(Y,Y) \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}$$

$$\overline{\underset{\sim}{X}}_n = \frac{1}{n} \sum_{i=1}^{n} \underset{\sim}{X}_i = \frac{1}{n} \sum_{i=1}^{n} \begin{bmatrix} X_i \\ Y_i \end{bmatrix} = \begin{bmatrix} \bar{X}_n \\ \bar{Y}_n \end{bmatrix}$$

$$\underset{\sim}{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

$$\sqrt{n}\left(\begin{bmatrix} \bar{X}_n \\ \bar{Y}_n \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \right) \xrightarrow{D} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} \right)$$

Theorem (Multivariate delta method)

Let $\mathbf{Y}_n = (Y_{n1}, \dots, Y_{nd})^T$, $n \geq 1$, *be a sequence of rvecs and* $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)^T$ *be a vector of real numbers such that*

$$\sqrt{n}(\mathbf{Y}_n - \boldsymbol{\theta}) \xrightarrow{D} \text{Normal}(\mathbf{0}, \boldsymbol{\Sigma}) \quad \text{as } n \to \infty.$$

Then for any function $g : \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\underset{d\times 1}{\dot{\mathbf{g}}(\boldsymbol{\theta})} = \left[\frac{\partial g(\boldsymbol{\theta})}{\partial \theta_1}, \dots, \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_d}\right]^T,$$

exists and is not equal to zero, we have

$$\sqrt{n}[g(\mathbf{Y}_n) - g(\boldsymbol{\theta})] \xrightarrow{D} \text{Normal}\,(0, \underbrace{[\dot{\mathbf{g}}(\boldsymbol{\theta})]^T}_{(d\times 1)^T} \underbrace{\boldsymbol{\Sigma}}_{d\times d} \underbrace{[\dot{\mathbf{g}}(\boldsymbol{\theta})]}_{d\times 1}) \quad \text{as } n \to \infty.$$

"Sandwich" form

Univariate (d=1).

$$\sqrt{n}\left(g(Y_n) - g(\theta)\right) \xrightarrow{D} N\left(0,\ g'(\theta)\,\sigma^2\, g'(\theta)\right)$$

$$\sqrt{n}\left(\begin{bmatrix}\bar{X}_n\\ \bar{Y}_n\end{bmatrix}-\begin{bmatrix}\mu_X\\ \mu_Y\end{bmatrix}\right)\xrightarrow{D} N\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\sigma_X^2 & \sigma_{XY}\\ \sigma_{XY} & \sigma_Y^2\end{bmatrix}\right)$$

Exercise: Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be independent copies of (X, Y), where

$$\mathbb{E}X = \mu_X, \quad \mathbb{E}Y = \mu_Y, \quad \operatorname{Var} X = \sigma_X^2, \quad \operatorname{Var} Y = \sigma_Y^2, \quad \operatorname{Cov}(X, Y) = \sigma_{XY}.$$

Give the asymptotic distribution of $\sqrt{n}(\bar{X}_n/\bar{Y}_n - \mu_X/\mu_Y)$ as $n \to \infty$.

$$\sqrt{n}\left(g\left(\begin{bmatrix}\bar{X}_n\\ \bar{Y}_n\end{bmatrix}\right)-g\left(\begin{bmatrix}\mu_X\\ \mu_Y\end{bmatrix}\right)\right)\longrightarrow N\Big(\underset{\text{scalar}}{0},\ \underbrace{\dot{g}\left(\begin{bmatrix}\mu_X\\ \mu_Y\end{bmatrix}\right)^T \Sigma\, \dot{g}\left(\begin{bmatrix}\mu_X\\ \mu_Y\end{bmatrix}\right)}\Big)$$

$$g\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\frac{x}{y}$$

$$\dot{g}\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}\frac{\partial}{\partial x} g\left(\begin{bmatrix}x\\y\end{bmatrix}\right) \\[2ex] \frac{\partial}{\partial y} g\left(\begin{bmatrix}x\\y\end{bmatrix}\right)\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x}\frac{x}{y} \\[2ex] \frac{\partial}{\partial y}\frac{x}{y}\end{bmatrix} = \begin{bmatrix}\frac{1}{y} \\[2ex] -\frac{x}{y^2}\end{bmatrix}$$

$$\dot{g}\left(\begin{bmatrix}\mu_x\\\mu_y\end{bmatrix}\right) = \begin{bmatrix}\frac{1}{\mu_y} \\[2ex] -\frac{\mu_x}{\mu_y^2}\end{bmatrix}$$

$$\dot{g}\left(\begin{bmatrix}\mu_x\\\mu_y\end{bmatrix}\right)^T \Sigma \, \dot{g}\left(\begin{bmatrix}\mu_x\\\mu_y\end{bmatrix}\right) = \begin{bmatrix}\frac{1}{\mu_y} \\[2ex] -\frac{\mu_x}{\mu_y^2}\end{bmatrix}^T \begin{bmatrix}\sigma_x^2 & \sigma_{xy} \\[2ex] \sigma_{xy} & \sigma_y^2\end{bmatrix} \begin{bmatrix}\frac{1}{\mu_y} \\[2ex] -\frac{\mu_x}{\mu_y^2}\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}\frac{1}{\mu_y} & -\frac{\mu_x}{\mu_y^2}\end{bmatrix} \begin{bmatrix}\sigma_x^2 & \sigma_{xy} \\[2ex] \sigma_{xy} & \sigma_y^2\end{bmatrix}}_{1 \times 2} \begin{bmatrix}\frac{1}{\mu_y} \\[2ex] -\frac{\mu_x}{\mu_y^2}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\sigma_x^2}{\mu_y} - \frac{\mu_x}{\mu_y^2}\sigma_{xy}, & \frac{1}{\mu_y}\sigma_{xy} - \frac{\mu_x}{\mu_y^2}\sigma_y^2\end{bmatrix} \begin{bmatrix}\frac{1}{\mu_y} \\[2ex] -\frac{\mu_x}{\mu_y^2}\end{bmatrix}$$

$$= \frac{\sigma_x^2}{\mu_y^2} - \frac{\mu_x \sigma_{xy}}{\mu_y^3} - \frac{\mu_x \sigma_{xy}}{\mu_y^3} + \frac{\mu_x^2}{\mu_y^2}\sigma_y^2$$

$$= \frac{\sigma_x^2}{\mu_y^2} - 2\frac{\mu_x \sigma_{xy}}{\mu_y^3} + \frac{\mu_x^2}{\mu_y^2}\sigma_y^2$$

$\Rightarrow$

$$\sqrt{n}\left(\frac{\bar{X}_n}{\bar{Y}_n} - \frac{\mu_x}{\mu_y}\right) \xrightarrow{D} N\left(0, \frac{\sigma_x^2}{\mu_y^2} - 2\frac{\mu_x \sigma_{xy}}{\mu_y^3} + \frac{\mu_x^2}{\mu_y^2}\sigma_y^2\right).$$

Exercise: Let $X \sim \text{Binomial}(n, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$, with $X \perp\!\!\!\perp Y$.

- Let $\hat{p}_1 = X/n$ and $\hat{p}_2 = Y/n$ and show that $\sqrt{n}\left(\begin{bmatrix}\hat{p}_1\\ \hat{p}_2\end{bmatrix} - \begin{bmatrix}p_1\\ p_2\end{bmatrix}\right) \xrightarrow{D} N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}p_1(1-p_1) & 0\\ 0 & p_2(1-p_2)\end{bmatrix}\right)$

$$\sqrt{n}\left[\frac{\hat{p}_1/(1-\hat{p}_1)}{\hat{p}_2/(1-\hat{p}_2)} - \frac{p_1/(1-p_1)}{p_2/(1-p_2)}\right] \xrightarrow{D} \text{Normal}(0, \vartheta) \quad \text{as } n \to \infty,$$

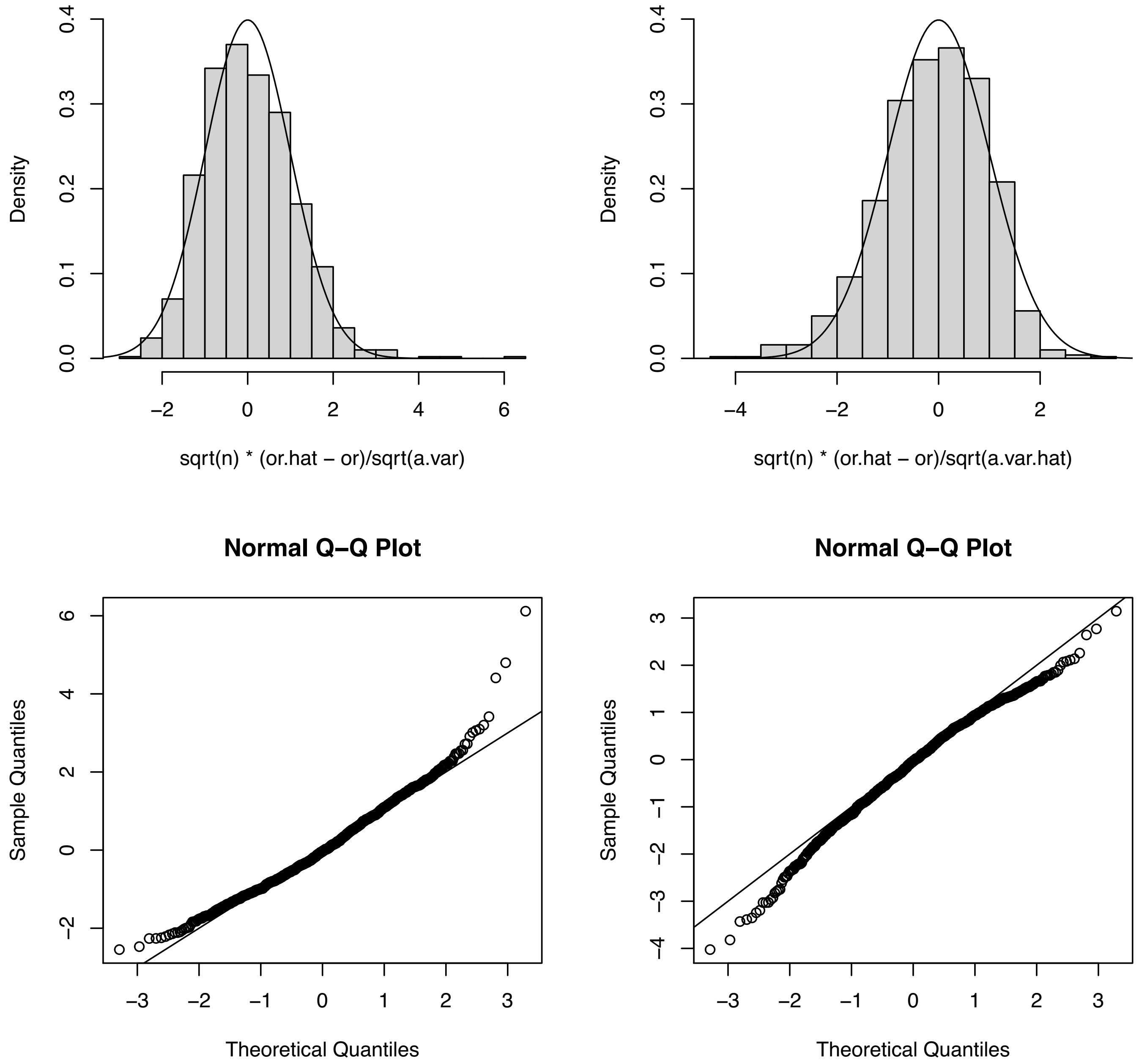
$$g\left(\begin{pmatrix}p_1\\p_2\end{pmatrix}\right) = \frac{p_1(1-p_1)}{p_2(1-p_2)}$$

where

$$\vartheta = \frac{p_1/(1-p_1)}{p_2/(1-p_2)}\left[\left(\frac{1-p_2}{p_2}\right)\frac{1}{(1-p_1)^2} + \left(\frac{p_1}{1-p_1}\right)\frac{1}{p_2^2}\right].$$

- Consider

$$\sqrt{n}\left(\overbrace{\log\left(\frac{\hat{p}_1(1-\hat{p}_1)}{\hat{p}_2(1-\hat{p}_2)}\right)}^{\text{log-odds}} - \log\left(\frac{p_1(1-p_1)}{p_2(1-p_2)}\right)\right) \xrightarrow{D} N\left(0, \frac{1}{p_1(1-p_1)} + \frac{1}{p_2(1-p_2)}\right)$$


Density
sqrt(n) * (or.hat – or)/sqrt(a.var)
Density
sqrt(n) * (or.hat – or)/sqrt(a.var.hat)
Normal Q–Q Plot
Sample Quantiles
Theoretical Quantiles
Normal Q–Q Plot
Sample Quantiles
Theoretical Quantiles