STAT 712 fa 2021 Exam 1

1. Show that Boole's inequality, which is $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$ for any events $A_1, \ldots, A_n$, implies

$$P(\cap_{i=1}^n A_i) \geq \sum_{i=1}^n P(A_i) - (n-1)$$

for any events $A_1, \ldots, A_n$.

2. There are two bags of marbles such that bag i has N_i marbles, M_i of which are red, for $i = 1, 2$. You will select one of the bags and grab K marbles from it, selecting bag i with probability p_i, $i = 1, 2$. Assume that $K < \min\{N_1 - M_1, M_1, N_2 - M_2, M_2\}$, so that it is possible for you to grab all red or all non-red marbles. Let X represent the number of red marbles you grab.

   (a) Given that you draw from bag 1, give an expression for the probability of $X = x$, $x = 0, \ldots, K$.

   (b) Give an expression for the probability of $X = x$, $x = 0, \ldots, K$.

   (c) Given that you observe $X = x$ for some $x = 0, \ldots, K$, give an expression for the probability that you drew from bag 1.

3. Let X be a continuous random variable with pdf given by

$$f_X(x) = \frac{1}{\beta} e^{-(x-c)/\beta} \cdot \mathbf{1}(x > c)$$

for some $\beta > 0$ and $c \in \mathbb{R}$.

(a) Explain in detail how to generate a realization of X starting with a Uniform$(0, 1)$ random variable.

(b) Give the moment generating function of X.

(c) Give $\mathbb{E}X$.

4. Let $X \sim f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}(0 < x < 1)$, for some $\alpha > 0$ and $\beta > 0$ and let $Y = \log(\frac{X}{1-X})$.

   (a) Give the pdf f_Y of the random variable Y. Be sure to give the support of Y.

   (b) (5 bonus points) Show that the mgf of Y is given by $M_Y(t) = \frac{\Gamma(\alpha+t)\Gamma(\beta-t)}{\Gamma(\alpha)\Gamma(\beta)}$ and give necessary restrictions on t.

   (c) Denote by $\Gamma'(\cdot)$ the first derivative of the gamma function $\Gamma(\cdot)$. Give an expression for $\mathbb{E}Y$.