# STAT 712 fa 2021 Exam 1

1. Show that Boole's inequality, which is $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$ for any events $A_1, \ldots, A_n$, implies

$$P(\cap_{i=1}^n A_i) \geq \sum_{i=1}^n P(A_i) - (n-1)$$

for any events $A_1, \ldots, A_n$.

**Solution:** By Boole's inequality and De Morgan's laws, we can write

$$\sum_{i=1}^n P(A_i^c) \geq P(\cup_{i=1}^n A_i^c) = P((\cap_{i=1}^n A_i)^c) = 1 - P(\cap_{i=1}^n A_i).$$

We can rearrange this to get

$$P(\cap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n P(A_i^c) = 1 - \sum_{i=1}^n [1 - P(A_i)] = \sum_{i=1}^n P(A_i) - (n-1).$$

2. There are two bags of marbles such that bag $i$ has $N_i$ marbles, $M_i$ of which are red, for $i = 1, 2$. You will select one of the bags and grab $K$ marbles from it, selecting bag $i$ with probability $p_i$, $i = 1, 2$. Assume that $K < \min\{N_1 - M_1, M_1, N_2 - M_2, M_2\}$, so that it is possible for you to grab all red or all non-red marbles. Let $X$ represent the number of red marbles you grab.

   (a) Given that you draw from bag 1, give an expression for the probability of $X = x$, $x = 0, \ldots, K$.

   **Solution:** We have

$$P(X = x|\text{bag } 1) = \frac{\binom{M_1}{x}\binom{N_1-M_1}{K-x}}{\binom{N_1}{K}} \qquad \text{for } x = 0, \ldots, K.$$

   (b) Give an expression for the probability of $X = x$, $x = 0, \ldots, K$.

   **Solution:** For all $x = 0, \ldots, K$, we have

$$\begin{aligned} P(X = x) &= P(X = x \cap \text{bag } 1) + P(X = x \cap \text{bag } 2) \\ &= P(X = x|\text{bag } 1)P(\text{bag } 1) + P(X = x|\text{bag } 2)P(\text{bag } 2) \\ &= \frac{\binom{M_1}{x}\binom{N_1-M_1}{K-x}}{\binom{N_1}{K}} \cdot p_1 + \frac{\binom{M_2}{x}\binom{N_2-M_2}{K-x}}{\binom{N_2}{K}} \cdot p_2. \end{aligned}$$

   (c) Given that you observe $X = x$ for some $x = 0, \ldots, K$, give an expression for the probability that you drew from bag 1.

   **Solution:** We can use Bayes' rule to write

$$\begin{aligned} P(\text{bag } 1|X = x) &= \frac{P(X = x|\text{bag } 1)P(\text{bag } 1)}{P(X = x|\text{bag } 1)P(\text{bag } 1) + P(X = x|\text{bag } 2)P(\text{bag } 2)} \\ &= \frac{\frac{\binom{M_1}{x}\binom{N_1-M_1}{K-x}}{\binom{N_1}{K}} \cdot p_1}{\frac{\binom{M_1}{x}\binom{N_1-M_1}{K-x}}{\binom{N_1}{K}} \cdot p_1 + \frac{\binom{M_2}{x}\binom{N_2-M_2}{K-x}}{\binom{N_2}{K}} \cdot p_2}. \end{aligned}$$

3. Let $X$ be a continuous random variable with pdf given by

$$f_X(x) = \frac{1}{\beta} e^{-(x-c)/\beta} \cdot \mathbf{1}(x > c)$$

for some $\beta > 0$ and $c \in \mathbb{R}$.

(a) Explain in detail how to generate a realization of $X$ starting with a Uniform$(0, 1)$ random variable.

**Solution:** We can generate a realization of $X$ by passing a Uniform$(0, 1)$ realization through its quantile function. For $x > c$, The cdf of $X$ is given by

$$F_X(x) = \int_c^x \frac{1}{\beta} e^{-(t-c)/\beta} dt = \cdots = 1 - e^{-(x-c)/\beta}.$$

Since this is a monotone function over the support of $X$, the quantile function of $X$ is the inverse of $F_X$. We find this by writing

$$u = 1 - e^{-(x-c)/\beta} \iff x = c - \beta \log(1 - u).$$

So we can generate $U \sim$ Uniform$(0, 1)$ and then set

$$X = c - \beta \log(1 - U).$$

(b) Give the moment generating function of $X$.

**Solution:** We have

$$\mathbb{E}e^{tX} = \int_c^\infty e^{tx} \cdot \frac{1}{\beta} e^{-(x-c)/\beta} dx = \frac{e^{ct}}{1 - \beta t},$$

provided $t < 1/\beta$.

(c) Give $\mathbb{E}X$.

**Solution:** We have

$$\mathbb{E}X = \int_c^\infty x \cdot \frac{1}{\beta} e^{-(x-c)/\beta} dx = \int_0^\infty (u + c) \cdot \frac{1}{\beta} e^{-u/\beta} = \beta + c.$$

4. Let $X \sim f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}(0 < x < 1)$, for some $\alpha > 0$ and $\beta > 0$ and let $Y = \log(\frac{X}{1-X})$.

(a) Give the pdf $f_Y$ of the random variable $Y$. Be sure to give the support of $Y$.

**Solution:** First, we note that the support of $Y$ is $\mathcal{Y} = (-\infty, \infty)$. Since the function is monotone, we can use the transformation method. We have

$$y = \log(x/(1-x)) =: g(x) \iff x = (1+e^{-y})^{-1} =: g^{-1}(y), \quad \frac{d}{dy} g^{-1}(y) = e^{-y}(1+e^{-y})^{-2},$$

so that

$$\begin{aligned} f_Y(y) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{1}{1+e^{-y}}\right)^{\alpha-1} \left(1 - \frac{1}{1+e^{-y}}\right)^{\beta-1} \left|\frac{e^{-y}}{(1+e^{-y})^2}\right| \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{1}{1+e^{-y}}\right)^{\alpha} \left(\frac{e^{-y}}{1+e^{-y}}\right)^{\beta} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{e^{-y\beta}}{(1+e^{-y})^{\alpha+\beta}} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{e^{y}}{1+e^{y}}\right)^{\alpha} \left(\frac{1}{1+e^{y}}\right)^{\beta} \end{aligned}$$

for all $y \in \mathbb{R}$.

(b) (5 bonus points) Show that the mgf of $Y$ is given by $M_Y(t) = \frac{\Gamma(\alpha+t)\Gamma(\beta-t)}{\Gamma(\alpha)\Gamma(\beta)}$ and give necessary restrictions on $t$.

**Solution:** We write

$$\begin{aligned} M_Y(t) &= \mathbb{E}e^{tY} \\ &= \int_{-\infty}^{\infty} e^{ty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{e^{-y\beta}}{(1+e^{-y})^{\alpha+\beta}} dy \\ &= \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{e^{-y(\beta-t)}}{(1+e^{-y})^{\alpha+\beta}} dy \\ &= \frac{\Gamma(\alpha+t)\Gamma(\beta-t)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+t)\Gamma(\beta-t)} \frac{e^{-y(\beta-t)}}{(1+e^{-y})^{(\alpha+t)+(\beta-t)}} dy}_{=1 \text{ if } \beta-t>0 \text{ and } \alpha+t>0}. \end{aligned}$$

The integral in the above is equal to 1 provided $\beta - t > 0$ and $\alpha + t > 0$, so we have

$$M_Y(t) = \frac{\Gamma(\alpha+t)\Gamma(\beta-t)}{\Gamma(\alpha)\Gamma(\beta)} \quad \text{for } -\alpha < t < \beta.$$

(c) Denote by $\Gamma'(\cdot)$ the first derivative of the gamma function $\Gamma(\cdot)$. Give an expression for $\mathbb{E}Y$.

**Solution:** We have

$$\mathbb{E}Y = \frac{d}{dt}\frac{\Gamma(\alpha+t)\Gamma(\beta-t)}{\Gamma(\alpha)\Gamma(\beta)}\bigg|_{t=0} = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\beta)}{\Gamma(\beta)}.$$