STAT 712 fa 2021 Exam 2

1. Let $X_1 \sim \text{Exponential}(\beta_1)$ and $X_2 \sim \text{Exponential}(\beta_2)$ be independent random variables and consider $R = X_1/X_2$ and $U = X_2$.

   (a) Give the joint pdf of random variable pair (R, U).

   The joint pdf of the random variable pair (X_1, X_2) is given by

   $$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\beta_1}e^{-x_1/\beta_1}\frac{1}{\beta_2}e^{-x_2/\beta_2} \cdot \mathbf{1}(x_1 > 0, x_2 > 0),$$

   and since $(X_1, X_2) \in (0,\infty) \times (0,\infty)$, we see that $(R, U) \in (0,\infty) \times (0,\infty)$. Now, we have

   $$\begin{array}{l} r = x_1/x_2 =: g_1(x_1, x_2) \\ u = x_2 =: g_2(x_1, x_2) \end{array} \iff \begin{array}{l} x_1 = ur =: g_1^{-1}(u, r) \\ x_2 = u =: g_2^{-1}(u, r) \end{array}$$

   with Jacobian

   $$J(x, y) = \begin{vmatrix} \frac{d}{du}ur & \frac{d}{dr}ur \\ \frac{d}{du}u & \frac{d}{dr}u \end{vmatrix} = \begin{vmatrix} r & u \\ 1 & 0 \end{vmatrix} = -u.$$

   The joint pdf of the random variable pair (R, U) is given by

   $$\begin{aligned} f_{R,U}(r, u) &= \frac{1}{\beta_1}e^{-(ur)/\beta_1}\frac{1}{\beta_2}e^{-u/\beta_2} \cdot u \\ &= \frac{1}{\beta_1\beta_2} \cdot u \cdot \exp\left[-u\left(\frac{r}{\beta_1} + \frac{1}{\beta_2}\right)\right] \end{aligned}$$

   for $r > 0$ and $u > 0$.

   (b) State whether R and U are independent and explain how you know.

   Since we cannot factorize the joint pdf $f_{R,U}(r, u)$ of (R, U) into the product of a function of only r and a function of only u, the random variables R and U are *not* independent.

   (c) Give the marginal pdf of R.

The marginal pdf of R is given by

$$f_R(r) = \int_0^\infty \frac{1}{\beta_1\beta_2} \cdot u \cdot \exp\left[-u\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)\right] du$$
$$= \frac{\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}}{\beta_1\beta_2} \int_0^\infty \frac{u}{\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}} \cdot \exp\left[-u/\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}\right] du$$
$$= \frac{\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}}{\beta_1\beta_2} \cdot \left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}$$
$$= \frac{\beta_1\beta_2}{(\beta_1+\beta_2 r)^2}$$

for $r > 0$.

(d) Give the conditional pdf of $U|R = r$.

For any $r > 0$, the conditional pdf of $U|R = r$ is given by

$$f(u|r) = \frac{f_{R,U}(r,u)}{f_R(r)}$$
$$= \frac{\frac{1}{\beta_1\beta_2} \cdot u \cdot \exp\left[-u\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)\right]}{\frac{\beta_1\beta_2}{(\beta_2+\beta_1 r)^2}}$$
$$= \frac{1}{\Gamma(2)\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)^{-2}} \cdot u^{2-1} \cdot \exp\left[-u/\left(\frac{r}{\beta_1}+\frac{1}{\beta_2}\right)^{-1}\right],$$

which we recognize as the pdf of the Gamma$(2, (r/\beta_1 + 1/\beta_2)^{-1})$ distribution.

2. Consider the pair of random variables (X, Y) arising from the hierarchical model

$$\begin{aligned} Y|X &\sim \text{Normal}(X, 1) \\ X &\sim p_X(x) = (1/2) \cdot \mathbf{1}(x = 1) + (1/2) \cdot \mathbf{1}(x = -1). \end{aligned}$$

(a) Give the mean and variance of Y.

We can quickly find that $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$, so that $\operatorname{Var} X = 1$. Then we have

$$\begin{aligned} \mathbb{E}Y &= \mathbb{E}\left(\mathbb{E}[Y|X]\right) = \mathbb{E}(X) = 0 \\ \operatorname{Var} Y &= \operatorname{Var}\left(\mathbb{E}[Y|X]\right) + \mathbb{E}\left(\operatorname{Var}[Y|X]\right) = \operatorname{Var}(X) + \mathbb{E}(1) = 1 + 1 = 2. \end{aligned}$$

(b) Give the marginal pdf of Y.

The marginal pdf of Y is given by

$$\begin{aligned} f_Y(y) &= \sum_{x \in \{-1,1\}} p_X(x) \cdot \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2} \\ &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} + \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-(y+1)^2/2}. \end{aligned}$$

(c) Give the covariance of X and Y.

We have

$$\operatorname{Cov}(X, Y) = \mathbb{E}XY - \underbrace{\mathbb{E}X}_{=0} \mathbb{E}Y = \mathbb{E}\left(\mathbb{E}[XY|X]\right) = \mathbb{E}\left(X \cdot \mathbb{E}[Y|X]\right) = \mathbb{E}\left(X \cdot X\right) = 1.$$

(d) Give the correlation of X and Y.

We have

$$\operatorname{Corr}(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var} X}\sqrt{\operatorname{Var} Y}} = \frac{1}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

3. Let $Z_1, \ldots, Z_8 \stackrel{\text{ind}}{\sim} \text{Normal}(0,1)$. Use $Z_1, \ldots, Z_8$ to construct a random variable having the

(a) $\text{Normal}(0, 1/8)$ distribution.

We have $(1/8)(Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8) \sim \text{Normal}(0, 1/8)$.

(b) χ^2_4 distribution.

We have $Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \sim \chi^2_4$.

(c) t_5 distribution.

We have
$$\frac{Z_1}{\sqrt{(Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2 + Z_6^2)/5}} \sim t_5.$$

(d) $F_{4,4}$ distribution.

We have
$$\frac{(Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2)/4}{(Z_5^2 + Z_6^2 + Z_7^2 + Z_8^2)/4} \sim F_{4,4}.$$

4. Let $X_1, X_2, \ldots$ be independent random variables such that

$$\log\left(\mathbb{E}\exp(tX_i)\right) = -(1/2)^i \log(1-t) \qquad \text{for } t < 1$$

for $i = 1, 2, \ldots$ Now, let $Y_n = \sum_{i=1}^n X_i$ for $n \geq 1$.

(a) Give the moment generating function $M_{Y_n}(t)$ of Y_n.

We see that the mgf of X_i is

$$M_{X_i}(t) = (1-t)^{-(1/2)^i} \qquad \text{for } t < 1, \quad i = 1, 2, \ldots,$$

so we have

$$M_{Y_n}(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n (1-t)^{-(1/2)^i} = (1-t)^{\sum_{i=1}^n (1/2)^i}.$$

(b) Give the limit as $n \to \infty$ of $M_{Y_n}(t)$ and identify the distribution to which it belongs.

Noting that

$$\lim_{n\to\infty} \sum_{i=1}^n (1/2)^i = 1,$$

we see that $\lim_{n\to\infty} M_{Y_n}(t) = (1-t)^{-1}$ for all $t < 1$, which is the moment generating function of the Exponential(1) distribution.