

## STAT 712 fa 2021 Final Exam

1. Let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} f_X(x) = e^{-(x-\mu)} \mathbf{1}(x > \mu)$  and let  $X_{(1)} < \dots < X_{(n)}$  be the order statistics.
  - (a) Give the pdf of  $X_{(1)}$ .
  - (b) Give the value to which  $X_{(1)}$  converges in probability; establish the convergence.
  - (c) Give the value to which  $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$  converges in probability; establish the convergence.

2. Let the random variable pair  $(X, Y)$  have joint density given by

$$f_{X,Y}(x, y) = \frac{6}{c} \exp\left[-\frac{y}{cx}\right] (1-x) \cdot \mathbf{1}(0 < x < 1, y > 0)$$

for some  $c > 0$ .

- (a) Find the marginal pdf  $f_X$  of  $X$ .
- (b) Find the conditional pdf  $f(y|x)$  of  $Y$  given  $X = x$ .
- (c) Give  $\mathbb{E}Y$  and  $\text{Var } Y$ .

3. Let  $Y_1, \dots, Y_n \stackrel{\text{ind}}{\sim} f_Y(y) = 3y^2 \cdot \mathbf{1}(0 < y < 1)$ .

(a) Find the values  $a$  and  $b$  such that

$$\sqrt{n} \left( n^{-1} \sum_{i=1}^n \frac{1}{Y_i} - a \right) \xrightarrow{D} \text{Normal}(0, b) \quad \text{as } n \rightarrow \infty.$$

(b) Find the values  $c$  and  $d$  such that

$$\sqrt{n} \left( \frac{1}{\bar{Y}_n} - c \right) \xrightarrow{D} \text{Normal}(0, d) \quad \text{as } n \rightarrow \infty.$$

4. For each  $n \geq 1$ , let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} f_X(x) = \alpha x^{-(\alpha+1)} \mathbf{1}(x \geq 1)$  for some  $\alpha > 0$ , and set  $Y_n = n^{-1/\alpha} X_{(n)}$ .
- (a) Give sequences  $a_n$  and  $b_n$  of real numbers such that  $a_n(\bar{X}_n - b_n) \xrightarrow{D} Z \sim \text{Normal}(0, 1)$ . Discuss any restrictions needed on the value of  $\alpha$ .
- (b) Give the cdf  $F_Y$  such that  $Y_n \xrightarrow{D} Y \sim F_Y$  as  $n \rightarrow \infty$ .