STAT 712 fa 2021 Final Exam

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x) = e^{-(x-\mu)}\mathbf{1}(x > \mu)$ and let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics.

   (a) Give the pdf of $X_{(1)}$.

   > The cdf corresponding to the pdf f_X is given by
   >
   > $$F_X(x) = 1 - e^{-(x-\mu)}$$
   >
   > for $x > \mu$, so we have
   >
   > $$f_{X_{(1)}}(x) = n[1 - (1 - e^{-(x-\mu)})]^{n-1}e^{-(x-\mu)} = ne^{-n(x-\mu)}$$
   >
   > for $x > \mu$.

   (b) Give the value to which $X_{(1)}$ converges in probability; establish the convergence.

   > We have $X_{(1)} \xrightarrow{p} \mu$, since, for any $\varepsilon > 0$, we have
   >
   > $$\begin{aligned} P(|X_{(1)} - \mu| < \varepsilon) &= P(\mu - \varepsilon < X_{(1)} < \mu + \varepsilon) \\ &= P(\mu < X_{(1)} < \mu + \varepsilon) \\ &= \int_{\mu}^{\mu+\varepsilon} ne^{-n(x-\mu)}dx \\ &= 1 - e^{-n\varepsilon} \\ &\to 1 \end{aligned}$$
   >
   > as $n \to \infty$.

   (c) Give the value to which $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ converges in probability; establish the convergence.

   > By the WLLN, $\bar{X}_n \xrightarrow{p} \mathbb{E}X_1$ provided $\operatorname{Var} X_1 < \infty$. We have $\mathbb{E}X_1 = \mu + 1$ and $\operatorname{Var} X_1 = 1$ (do integrals or note that the distribution is a shifted exponential), so $\bar{X}_n \xrightarrow{p} \mu + 1$.

2. Let the random variable pair (X, Y) have joint density given by

$$f_{X,Y}(x, y) = \frac{6}{c} \exp\left[-\frac{y}{cx}\right](1 - x) \cdot \mathbf{1}(0 < x < 1, y > 0)$$

for some $c > 0$.

(a) Find the marginal pdf f_X of X.

We have

$$f_X(x) = \int_0^\infty \frac{6}{c} \exp\left[-\frac{y}{cx}\right](1 - x) = 6x(1 - x),$$

so we write

$$f_X(x) = 6x(1 - x)\mathbf{1}(0 < x < 1).$$

(b) Find the conditional pdf $f(y|x)$ of Y given $X = x$.

For any $x \in (0, 1)$, we have

$$f(y|x) = \frac{1}{cx} \exp\left[-\frac{y}{cx}\right] \mathbf{1}(y > 0).$$

(c) Give $\mathbb{E}Y$ and $\operatorname{Var} Y$.

Noting that $Y|X = x \sim \text{Exponential}(cx)$ and that

$$\mathbb{E}X = \frac{1}{2} \quad \text{and } \operatorname{Var} X = \frac{1}{20},$$

we have

$$\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|X]) = \mathbb{E}(cX) = \frac{c}{2}$$

and

$$\operatorname{Var} Y = \operatorname{Var}(\mathbb{E}[Y|X]) + \mathbb{E}(\operatorname{Var}[Y|X]) = \operatorname{Var}(cX) + \mathbb{E}(c^2X^2) = \frac{c^2}{20} + c^2\left[\frac{1}{20} + \left(\frac{1}{2}\right)^2\right] = c^2 \cdot \frac{7}{20}.$$

3. Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} f_Y(y) = 3y^2 \cdot \mathbf{1}(0 < y < 1)$.

(a) Find the values a and b such that

$$\sqrt{n}\left(n^{-1}\sum_{i=1}^{n}\frac{1}{Y_i} - a\right) \xrightarrow{\text{D}} \text{Normal}(0, b) \quad \text{as } n \to \infty.$$

Letting $U_1 = 1/Y_1$, we have

$$\mathbb{E}U_1 = \int_0^1 \frac{1}{y} \cdot 3y^2 dy = \int_0^1 3y dy = \frac{3}{2}y^2\Big|_0^1 = \frac{3}{2}.$$

and

$$\mathbb{E}U_1^2 = \int_0^1 \frac{1}{y^2} \cdot 3y^2 dy = \int_0^1 3dy = 3,$$

so that $\operatorname{Var} U_1 = 3 - (3/2)^2 = 3/4$. Therefore, the convergence in distribution holds for

$$a = \frac{3}{2} \quad \text{and} \quad b = \frac{3}{4}$$

by the central limit theorem.

(b) Find the values c and d such that

$$\sqrt{n}\left(\frac{1}{\bar{Y}_n} - c\right) \xrightarrow{\text{D}} \text{Normal}(0, d) \quad \text{as } n \to \infty.$$

We have

$$\mathbb{E}Y_1 = \int_0^1 y \cdot 3y^2 dy = \int_0^1 3y^3 dy = 3/4$$

and

$$\mathbb{E}Y_1^2 = \int_0^1 y^2 \cdot 3y^2 dy = \int_0^1 3y^4 dy = 3/5,$$

so that $\operatorname{Var} Y_1 = (3/5) - (3/4)^2 = 3/80$. So the central limit theorem gives

$$\sqrt{n}(\bar{Y}_n - 3/4) \xrightarrow{\text{D}} \text{Normal}(0, 3/80)$$

as $n \to \infty$. Now use the delta method with the function $g(z) = z^{-1}$, which has derivative $g'(z) = -1/z^2$. We obtain

$$\sqrt{n}((\bar{Y}_n)^{-1} - 4/3) \xrightarrow{\text{D}} \text{Normal}(0, [-1/(3/4)^2]^2 \cdot 3/80),$$

as $n \to \infty$, where the asymptotic variance simplifies to $[-1/(3/4)^2]^2 \cdot 3/80 = 16/135$. So the convergence in distribution holds for

$$c = \frac{4}{3} \quad \text{and} \quad d = \frac{16}{135}$$

4. For each $n \geq 1$, let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x) = \alpha x^{-(\alpha+1)} \mathbf{1}(x \geq 1)$ for some $\alpha > 0$, and set $Y_n = n^{-1/\alpha} X_{(n)}$.

(a) Give sequences a_n and b_n of real numbers such that $a_n(\bar{X}_n - b_n) \xrightarrow{\text{D}} Z \sim \text{Normal}(0,1)$. Discuss any restrictions needed on the value of α.

Provided $\alpha > 2$, we can work out that

$$\mathbb{E}X_1 = \frac{\alpha}{\alpha - 1} \quad \text{and} \quad \mathbb{E}X_1^2 = \frac{\alpha}{\alpha - 2},$$

so that

$$\text{Var}\, X_1 = \frac{\alpha}{\alpha - 2} - \left(\frac{\alpha}{\alpha - 1}\right)^2 = \frac{\alpha}{(\alpha - 2)(\alpha - 1)^2}.$$

Then the central limit theorem gives that

$$\sqrt{n \cdot \frac{(\alpha - 2)(\alpha - 1)^2}{\alpha}} \left(\bar{X}_n - \frac{\alpha}{\alpha - 1}\right) \xrightarrow{\text{D}} Z \sim \text{Normal}(0,1),$$

so the convergence holds under

$$a_n = \sqrt{n \cdot \frac{(\alpha - 2)(\alpha - 1)^2}{\alpha}} \quad \text{and} \quad b_n = \frac{\alpha}{\alpha - 1}.$$

(b) Give the cdf F_Y such that $Y_n \xrightarrow{\text{D}} Y \sim F_Y$ as $n \to \infty$.

First, the cdf of $X_{(n)}$ is given by $F_{X_{(n)}}(x) = (1 - x^{-\alpha})^n$, for $x \geq 1$. Next, note that $Y_n \in (n^{-1/\alpha}, \infty)$. Now, for any $y \geq n^{-1/\alpha}$, we have

$$\begin{aligned} F_{Y_n}(y) &= P(Y_n < y) \\ &= P(n^{-1/\alpha} X_{(n)} < y) \\ &= P(X_{(n)} < n^{1/\alpha} y) \\ &= (1 - (n^{1/\alpha} y)^{-\alpha})^n \\ &= (1 - y^{-\alpha}/n)^n \end{aligned}$$

From this we see that $F_{Y_n}(y) \to e^{-y^{-\alpha}}$ as $n \to \infty$ for any $y > 0$, so $Y_n \xrightarrow{\text{D}} Y \sim F_Y$, where

$$F_Y = \begin{cases} e^{-y^{-\alpha}} & y > 0 \\ 0 & y \leq 0. \end{cases}$$