STAT 712 fa 2022 Exam 1

1. Let A, B, and C be events such that A and B are independent with $P(A) = 1/2$ and $P(B) = 1/3$, and

$$P(C|A^c \cap B) = P(C|A \cap B^c) = P(C|A \cap B) = P(C^c|A^c \cap B^c) = 3/4.$$

(a) Give $P(C \cap A^c \cap B^c)$.

We have

$$\begin{aligned} P(C \cap A^c \cap B^c) &= P(C|A^c \cap B^c)P(A^c \cap B^c) \\ &= (1 - P(C^c|A^c \cap B^c))P(A^c)P(B^c) \\ &= (1 - 3/4) \cdot 1/2 \cdot 2/3 \\ &= 1/12. \end{aligned}$$

(b) Give $P(C \cap A^c)$.

We have

$$\begin{aligned} P(C \cap A^c) &= P(C \cap A^c \cap B) + P(C \cap A^c \cap B^c) \\ &= P(C|A^c \cap B)P(A^c \cap B) + 1/12 \\ &= 3/4 \cdot 1/2 \cdot 1/3 + 1/12 \\ &= 5/24. \end{aligned}$$

(c) Give $P(A|C)$.

We have

$$P(A|C) = \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|A^c)P(A^c)},$$

where

$$\begin{aligned} P(C|A) &= P(C|A \cup B) = 3/4 \\ P(C|A^c) &= P(C \cap A^c)/P(A^c) = (5/24)/(1/2) = 5/12. \end{aligned}$$

Plugging in these values gives

$$P(A|C) = \frac{3/4 \cdot 1/2}{3/4 \cdot 1/2 + 5/12 \cdot 1/2} = 9/14.$$

2. Let $X \sim f_X(x) = \alpha e^{\alpha x} e^{-e^{\alpha x}}$ for all $x \in \mathbb{R}$ for some $\alpha > 0$. Let $Y = e^{\alpha X}$.

(a) Give the pdf of Y. Make sure to define it for all $y \in \mathbb{R}$.

We first note that $\mathcal{Y} = (0, \infty)$. Now we have

$$y = e^{\alpha x} = g(x) \iff x = (\log y)/\alpha = g^{-1}(y), \quad \frac{d}{dy} g^{-1}(y) = \frac{1}{\alpha y},$$

so the transformation method gives

$$f_Y(y) = \alpha e^{\alpha (\log y)/\alpha} e^{-e^{\alpha(\log y)/\alpha}} \left| \frac{1}{\alpha y} \right| \cdot \mathbf{1}(y > 0) = e^{-y} \cdot \mathbf{1}(y > 0).$$

(b) Give the mgf of Y.

We have

$$\mathbb{E}e^{tY} = \int_0^\infty e^{ty} e^{-y} dy = \int_0^\infty e^{-y(1-t)} dy = -\frac{e^{-y(1-t)}}{1-t}\Big|_0^\infty = (1-t)^{-1},$$

provided $t < 1$. So

$$M_Y(t) = (1-t)^{-1} \quad \text{for } t < 1.$$

(c) Give $\mathbb{E}(Y - \mathbb{E}Y)^3$.

Noting that $\mathbb{E}Y^k = \int_0^\infty y^{(k+1)-1} e^{-y} = \Gamma(k)$, we have

$$\begin{aligned}
\mathbb{E}(Y - \mathbb{E}Y)^3 &= \mathbb{E}Y^3 - 3\mathbb{E}Y^2 \cdot \mathbb{E}Y + 3\mathbb{E}Y \cdot (\mathbb{E}Y)^2 - (\mathbb{E}Y)^3 \\
&= \Gamma(4) - 3\Gamma(3)\Gamma(2) + 3\Gamma(2)\Gamma(2)^2 - \Gamma(2)^3 \\
&= 6 - 3(2) + 3 - 1 \\
&= 2.
\end{aligned}$$

3. Consider the pdf given by

$$f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & 2 \leq x. \end{cases}$$

(a) For $X \sim f$, give $\mathbb{E}X$.

The pdf f is symmetric around $x = 1$, so $\mathbb{E}X = 1$.

(b) Give the cdf F corresponding to the pdf f. Make sure to define it for all $x \in \mathbb{R}$.

We have

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \leq x < 1 \\ 1 - (2 - x)^2/2, & 1 \leq x < 2 \\ 1, & 2 \leq x. \end{cases}$$

Drawing a picture of the pdf and computing the areas of the triangle-shaped regions is the simplest way to obtain this.

(c) Suppose $U \sim \text{Uniform}(0, 1)$. Explain how you would find a transformation g such that $X = g(U)$ has pdf f (you do not need to give the transformation).

We must invert the cdf and set $g = F^{-1}$. Then $X = F^{-1}(U)$ would be a random variable with cdf F. Setting $u = F(x)$ over each piece, we obtain

$$g(u) = \begin{cases} \sqrt{2u}, & 0 < u \leq 1/2 \\ 2 - \sqrt{2(1 - u)}, & 1/2 < u < 1. \end{cases}$$

4. Let $U \sim \text{Uniform}(0, 1)$ and let $V = 1 - U$. Show that U and V are identically distributed (have the same cdf).

We must show that the U and V have the same cdf. The cdf of U is given by $F_U(u) = u$ for $u \in (0, 1)$ and, for $v \in (0, 1)$, the cdf of V is given by

$$F_V(v) = P(V \leq v) = P(1 - U \leq v) = P(U \geq 1 - v) = 1 - P(U < 1 - v) = 1 - (1 - v) = v.$$

So U and V are identically distributed.