STAT 712 fa 2022 Exam 2

1. Let $T|U \sim \text{Normal}(cU, 1)$ and $U \sim \text{Normal}(0,1)$ for some $c \in \mathbb{R}$.

   (a) Give $\mathbb{E}T$ and $\operatorname{Var} T$.

   We have $\mathbb{E}T = \mathbb{E}(\mathbb{E}[T|U]) = \mathbb{E}(cU) = c\mathbb{E}U = 0$ and

   $$\operatorname{Var} T = \mathbb{E}(\operatorname{Var}[T|U]) + \operatorname{Var}(\mathbb{E}[T|U]) = \mathbb{E}(1) + \operatorname{Var}(cU) = 1 + c^2.$$

   (b) Give $\operatorname{Corr}(T, U)$.

   We have $\mathbb{E}TU = \mathbb{E}(\mathbb{E}[TU|U]) = \mathbb{E}(U\mathbb{E}[T|U]) = \mathbb{E}(U \cdot cU) = c\mathbb{E}U^2 = c$. Since $\mathbb{E}T = 0$ and $\mathbb{E}U = 0$ we have $\operatorname{Cov}(T, U) = \mathbb{E}TU = c$. From here we have

   $$\operatorname{Corr}(T, U) = \frac{c}{\sqrt{1} \cdot \sqrt{1+c^2}} = \frac{c}{\sqrt{1+c^2}}.$$

   (c) Derive the marginal pdf of T.

   We have

   $$\begin{aligned}
   f_T(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-cu)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \\
   &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2 - 2zcu + c^2u^2 + u^2}{2}\right) du \\
   &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2(1+c^2) - 2zcu}{2}\right) du \\
   &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2 - 2uzc/(1+c^2)}{2/(1+c^2)}\right) du \\
   &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} + \frac{z^2c^2}{2(1+c^2)}\right) \\
   &\quad \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2 - 2uzc/(1+c^2) + (zc/(1+c^2))^2}{2/(1+c^2)}\right) du \\
   &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+c^2}} \exp\left(-\frac{z^2}{2(1+c^2)}\right),
   \end{aligned}$$

   so that $Z \sim \text{Normal}(0, 1+c^2)$.

2. Let (X_1, X_2) have joint pdf given by

$$f(x_1, x_2) = \frac{1}{x_2\lambda} \exp\left[-\frac{x_1}{x_2\lambda}\right] \mathbf{1}(0 < x_2 < 1, x_1 > 0)$$

for some $\lambda > 0$.

(a) i. Give the marginal pdf of X_2.

For $x_2 \in (0, 1)$ we have

$$f_{X_2}(x_2) = \int_0^\infty \frac{1}{x_2\lambda} \exp\left[-\frac{x_1}{x_2\lambda}\right] dx_1 = 1,$$

so $X_2 \sim \text{Uniform}(0, 1)$.

ii. Give the conditional pdf of X_1 given $X_2 = x_2$.

Dividing $f(x_1, x_2)$ by the marginal pdf of X_2, we obtain

$$f(x_1|x_2) = \int_0^\infty \frac{1}{x_2\lambda} \exp\left[-\frac{x_1}{x_2\lambda}\right] \mathbf{1}(x_1 > 0),$$

so that $X_1|X_2 = x_2 \sim \text{Exponential}(x_2\lambda)$.

iii. Give $\mathbb{E}X_1$ and $\operatorname{Var} X_1$.

Noting that $\mathbb{E}[X_1|X_2] = X_2\lambda$ and $\mathbb{E}X_2 = 1/2$, the iterated expectation formula gives

$$\mathbb{E}X_1 = \mathbb{E}(\mathbb{E}[X_1|X_2]) = \mathbb{E}(X_2\lambda) = \lambda/2,$$

and using $\operatorname{Var}[X_1|X_2] = (X_2\lambda)^2$ and $\operatorname{Var} X_2 = 1/12$, the iterated variance formula gives

$$\begin{aligned}\operatorname{Var} X_1 &= \mathbb{E}(\operatorname{Var}[X_1|X_2]) + \operatorname{Var}(\mathbb{E}[X_1|X_2])) \\ &= \mathbb{E}(X_2^2\lambda^2) + \operatorname{Var}(X_2\lambda) \\ &= \lambda^2(1/3) + \lambda^2(1/12) \\ &= \lambda^2(5/12).\end{aligned}$$

(b) i. Find the joint pdf of $R = X_1/X_2$ and $V = X_2$.

Firstly, the joint support of (R, V) is $\{(r, u) : r > 0, v \in (0, 1)\}$. Now we have

$$\begin{matrix} r = x_1/x_2 =: g_1(x_1, x_2) \\ v = x_2 =: g_2(x_1, x_2) \end{matrix} \iff \begin{matrix} x_1 = rv =: g_1^{-1}(r, v) \\ x_2 = v =: g_2^{-1}(r, v) \end{matrix}$$

with Jacobian

$$J(r, v) = \begin{vmatrix} \frac{d}{dr}rv & \frac{d}{dv}rv \\ \frac{d}{dr}v & \frac{d}{dv}v \end{vmatrix} = \begin{vmatrix} v & r \\ 0 & 1 \end{vmatrix} = v.$$

So the joint pdf of (R, V) is given by

$$f(r, v) = \frac{1}{v\lambda} \exp\left[-\frac{rv}{v\lambda}\right] |v| = \frac{1}{\lambda} e^{-r/\lambda}$$

for $r > 0$ and $v \in (0, 1)$.

ii. State whether R and V are independent. Explain.

The random variables R and V are independent since we can factor their joint pdf as

$$f(r, v) = g(r)h(v), \quad g(r) = \frac{1}{\lambda} e^{-r/\lambda} \mathbf{1}(r > 0), \quad h(v) = \mathbf{1}(0 < v < 1).$$

iii. Find the marginal pdf of R.

For $r > 0$ the marginal pdf of R is given by

$$f_R(r) = \int_0^1 \frac{1}{\lambda} e^{-r/\lambda} dv = \frac{1}{\lambda} e^{-r/\lambda},$$

so $R \sim \text{Exponential}(\lambda)$.

3. Let Z_1, Z_2, Z_3, Z_4 be independent Normal$(0, 1)$ rvs.

(a) Give the distribution of each of the following:

i. $(Z_1 + Z_2)/(\sqrt{2}|Z_3|)$

We have $(Z_1 + Z_2)/(\sqrt{2}|Z_3|) \sim t_1$, since $(Z_1 + Z_2)/\sqrt{2} \sim$ Normal$(0, 1)$ and $|Z_3| = \sqrt{Z_3^2/1}$.

ii. $(1/6)(Z_1 - 2Z_2 + Z_3)^2$

We have $(1/6)(Z_1 - 2Z_2 + Z_3)^2 \sim \chi_1^2$, since $Z_1 - 2Z_2 + Z_3 \sim$ Normal$(0, 6)$, giving $(Z_1 - 2Z_2 + Z_3)/\sqrt{6} \sim$ Normal$(0, 1)$.

iii. $Z_1^2/(Z_2/\sqrt{2} - Z_3/\sqrt{2})^2$

We have $Z_1^2/(Z_2/\sqrt{2} - Z_3/\sqrt{2})^2 \sim F_{1,1}$.

iv. $(1/2)[(Z_1 - Z_2)^2 + (Z_1 + Z_2)^2]$

We have $(1/2)[(Z_1 - Z_2)^2 + (Z_1 + Z_2)^2] \sim \chi_2^2$, since $\text{Cov}(Z_1 - Z_2, Z_1 + Z_2) = 0$, implying independence between $(Z_1 - Z_2)^2/2$ and $(Z_1 + Z_2)^2/2$. Since these are both χ_1^2 random variables, their sum is a χ_2^2 random variable.

v. $(1/3)[(Z_1 - Z_2)^2 + (Z_2 - Z_3)^2 + (Z_1 - Z_3)^2]$

We have $(1/3)[(Z_1 - Z_2)^2 + (Z_2 - Z_3)^2 + (Z_1 - Z_3)^2] \sim \chi_2^2$, since

$$\frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (Z_i - Z_j)^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2$$

(showing this was a homework problem), so

$$\frac{1}{n} \sum_{i<j} (Z_i - Z_j)^2 = \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2 \sim \chi_{n-1}^2.$$

For $n = 3$ the right hand side is $(1/3)[(Z_1 - Z_2)^2 + (Z_2 - Z_3)^2 + (Z_1 - Z_3)^2]$. We can also show directly

$$\begin{aligned}
(1/3)[(Z_1 - Z_2)^2 + (Z_2 - Z_3)^2 + (Z_1 - Z_3)^2] &= (1/3)[2Z_1^2 + 2Z_2^2 + 2Z_3^2 - 2Z_1Z_2 - 2Z_2Z_3 - 2Z_1Z_3] \\
&= (1/3)[3(Z_1^2 + Z_2^2 + Z_3^2) - (Z_1 + Z_2 + Z_3)^2] \\
&= \textstyle\sum_{i=1}^{3} Z_i^2 - (3\bar{Z}_3)^2/3 \\
&= \textstyle\sum_{i=1}^{3} Z_i^2 - 3\bar{Z}_3^2 \\
&= \textstyle\sum_{i=1}^{3} (Z_i - \bar{Z}_3)^2,
\end{aligned}$$

which follows a χ_2^2 distribution.

(b) Give

i. $\text{Cov}(Z_1 + Z_2, Z_1 - Z_2)$

> We obtain $\text{Cov}(Z_1 + Z_2, Z_1 - Z_2) = 0$, since $1(1) + 1(-1) = 0$

ii. $\text{Cov}(Z_1 + Z_2 + Z_3 + Z_4, Z_1 - Z_2 + Z_3 - Z_4)$

> We obtain $\text{Cov}(Z_1+Z_2+Z_3+Z_4, Z_1-Z_2+Z_3-Z_4) = 0$, since $1(1)+1(-1)+1(1)+1(-1) = 0$