

STAT 712 fa 2022 Final Exam

1. For $n \geq 1$, let X_1, \dots, X_n be iid with mgf $M_X(t) = (1 - t/3)^{-\alpha}$ for $t < 3$ for some $\alpha > 0$.
 - (a) Find constants a and b such that $\sqrt{n}(\bar{X}_n - a) \xrightarrow{D} \text{Normal}(0, b)$ as $n \rightarrow \infty$.
 - (b) Find constants c and d such that $\sqrt{n}(\log \bar{X}_n - c) \xrightarrow{D} \text{Normal}(0, d)$ as $n \rightarrow \infty$.
 - (c) Find a function $\hat{\alpha}_n$ of X_1, \dots, X_n such that $\hat{\alpha}_n \xrightarrow{p} \alpha$. Prove the convergence.

2. For $n \geq 1$, let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Uniform}(0, 1/\theta)$, for some $\theta > 0$.

(a) Find the pdf of $Y_n = 1/X_{(n)}$.

Recall that the cdf of the largest order statistic is the population cdf raised to the power n .

(b) Show that $Y_n \xrightarrow{p} \theta$ as $n \rightarrow \infty$.

3. Let (X, Y) have joint pdf given by $f(x, y) = 2\beta x^{\beta-3} y^{-(\beta+1)} \mathbf{1}(1 < x < y)$ for some $\beta > 1$.
- (a) Find the marginal pdf of X .
 - (b) Give a transformation g such that $g(U) \stackrel{d}{=} X$, where $U \sim \text{Uniform}(0, 1)$.
 - (c) Find the conditional pdf of $Y|X = x$.
 - (d) Give $\mathbb{E}Y$.

4. Let (Y, D) be a pair of random variables such that

$$Y|D = d \sim f(y|d) = \frac{2}{d\pi} \frac{1}{1 + (y/d)^2} \cdot \mathbf{1}(-d < y < d)$$
$$D \sim f_D(d) = 2d \cdot \mathbf{1}(0 < d < 1).$$

- (a) Give $\text{Corr}(Y, D)$.
- (b) Write down an integral expression which will yield the marginal pdf of Y . Give the support of Y .
- (c) Let (U, V) be the random variables $U = Y/D$ and $V = D$. Find the joint pdf of (U, V) .
- (d) Give $\text{Cov}(U, V)$.