STAT 712 fa 2022 Final Exam

1. For $n \geq 1$, let $X_1, \ldots, X_n$ be iid with mgf $M_X(t) = (1 - t/3)^{-\alpha}$ for $t < 3$ for some $\alpha > 0$.

   (a) Find constants a and b such that $\sqrt{n}(\bar{X}_n - a) \xrightarrow{\text{D}} \text{Normal}(0, b)$ as $n \to \infty$.

   > For a rv X with mgf M_X, we have $\mathbb{E}X = \alpha/3$ and $\operatorname{Var} X = \alpha/9$. The central limit theorem therefore gives $\sqrt{n}(\bar{X}_n - \alpha/3) \xrightarrow{\text{D}} \text{Normal}(0, \alpha/9)$, so
   >
   > $$a = \alpha/3 \quad \text{and} \quad b = \alpha/9.$$

   (b) Find constants c and d such that $\sqrt{n}(\log \bar{X}_n - c) \xrightarrow{\text{D}} \text{Normal}(0, d)$ as $n \to \infty$.

   > Beginning with the result in the previous part and applying the delta method under the function $g(x) = \log x$, where $g'(x) = 1/x$, gives $\sqrt{n}(\log \bar{X}_n - \log(\alpha/3)) \xrightarrow{\text{D}} \text{Normal}(0, [1/(\alpha/3)]^2 \cdot \alpha/9)$, so that
   >
   > $$c = \log(\alpha/3) \quad \text{and} \quad d = [1/(\alpha/3)]^2 \cdot \alpha/9 = 1/\alpha.$$

   (c) Find a function $\hat{\alpha}_n$ of $X_1, \ldots, X_n$ such that $\hat{\alpha}_n \xrightarrow{p} \alpha$. Prove the convergence.

   > The WLLN gives $\bar{X}_n \xrightarrow{p} \alpha/3$, so we can set $\hat{\alpha}_n = 3\bar{X}_n$. To prove $\hat{\alpha}_n \xrightarrow{p} \alpha$ as $n \to \infty$ note that for each $\varepsilon > 0$ we have
   >
   > $$P(|\hat{\alpha}_n - \alpha| < \varepsilon) = P(|3\bar{X}_n - \alpha| < \varepsilon) = P(|\bar{X}_n - \alpha/3| < \varepsilon/3) \to 0$$
   >
   > as $n \to \infty$.

2. For $n \geq 1$, let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, 1/\theta)$, for some $\theta > 0$.

(a) Find the pdf of $Y_n = 1/X_{(n)}$.
Recall that the cdf of the largest order statistic is the population cdf raised to the power n.

The cdf of the Uniform$(0, 1/\theta)$ distribution is given by $F_X(x) = x\theta$ for $0 < x < 1/\theta$, so the cdf of $X_{(n)}$ is given by

$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x) = P(\text{all of } X_1, \ldots, X_n \text{ are} \leq x) = [F_X(x)]^n = (x\theta)^n$$

for $0 < x < 1/\theta$. The random variable $Y_n = 1/X_{(n)}$ has support on (θ, ∞), with cdf given by

$$P(Y_n \leq y) = P(1/X_{(n)} \leq y) = P(1/y \leq X_{(n)}) = 1 - P(X_{(n)} < 1/y) = 1 - (\theta/y)^n$$

for $\theta < y < \infty$. Now the pdf of Y_n is given by

$$f_{Y_n}(y) = \frac{d}{dy} F_{Y_n}(y) = \frac{n\theta^n}{y^{n+1}}$$

for $\theta < y < \infty$.

(b) Show that $Y_n \xrightarrow{p} \theta$ as $n \to \infty$.

For any $\varepsilon > 0$ we have

$$\begin{aligned} P(|Y_n - \theta| < \varepsilon) &= P(\theta - \varepsilon < Y_n < \theta + \varepsilon) \\ &= \int_{\theta-\varepsilon}^{\theta+\varepsilon} \frac{n\theta^n}{y^{n+1}} \mathbf{1}(y > \theta) dy \\ &= \int_{\theta}^{\theta+\varepsilon} \frac{n\theta^n}{y^{n+1}} dy \\ &= -\frac{\theta^n}{y^n}\Big|_{\theta}^{\theta+\varepsilon} \\ &= 1 - \left(\frac{\theta}{\theta+\varepsilon}\right)^n, \end{aligned}$$

which limits to 1 as $n \to \infty$, so $Y_n \xrightarrow{p} \theta$.

3. Let (X, Y) have joint pdf given by $f(x,y) = 2\beta x^{\beta-3} y^{-(\beta+1)} \mathbf{1}(1 < x < y)$ for some $\beta > 1$.

(a) Find the marginal pdf of X.

The support of X is $(1, \infty)$. For each $x \in (1, \infty)$ the marginal pdf of X is given by

$$\begin{aligned} f_X(x) &= \int_x^\infty 2\beta x^{\beta-3} y^{-(\beta+1)} dy \\ &= 2\beta x^{\beta-3} \left[\frac{y^{-(\beta+1)+1}}{-(\beta+1)+1} \right] \Big|_x^\infty \\ &= 2\beta x^{\beta-3} \left[0 - \frac{x^{-\beta}}{-\beta} \right] \\ &= 2x^{-3}. \end{aligned}$$

(b) Give a transformation g such that $g(U) \stackrel{d}{=} X$, where $U \sim \text{Uniform}(0,1)$.

The probability integral transform gives $Q_X(U) \stackrel{d}{=} X$, where Q_X is the quantile function of X. The cdf of X is given by

$$F_X(x) = \int_1^x 2t^{-3} dt = 2 \left[\frac{t^{-2}}{-2} \right] \Big|_1^x = 1 - \frac{1}{x^2}$$

for $x \in (1, \infty)$. We have

$$u = 1 - \frac{1}{x^2} \iff x = \sqrt{\frac{1}{1-u}},$$

so the quantile function of X is $Q_X(u) = \sqrt{1/(1-u)}$.

(c) Find the conditional pdf of $Y|X = x$.

The support of $Y|X = x$ is (x, ∞). For each $x \in (1, \infty)$, we have

$$f(y|x) = \frac{2\beta x^{\beta-3} y^{-(\beta+1)}}{2x^{-3}} \mathbf{1}(y > x) = \frac{\beta x^\beta}{y^{\beta+1}} \mathbf{1}(y > x).$$

(d) Give $\mathbb{E}Y$.

The iterated expectation formula gives $\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|X])$, where

$$\mathbb{E}[Y|X = x] = \int_x^\infty y \cdot \frac{\beta x^\beta}{y^{\beta+1}} dy = \beta x^\beta \left[\frac{y^{-\beta+1}}{-\beta+1} \right] \Big|_x^\infty = \beta x^\beta \left[0 - \frac{x^{-\beta+1}}{-\beta+1} \right] = \frac{x\beta}{\beta - 1}.$$

In addition

$$\mathbb{E}X = \int_1^\infty x \cdot 2x^{-3} dx = 2 \left[-x^{-1} \right] \Big|_1^\infty = 2,$$

so we have

$$\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|X]) = \mathbb{E}\left(\frac{X\beta}{\beta - 1}\right) = \frac{2\beta}{\beta - 1}.$$

4. Let (Y, D) be a pair of random variables such that

$$Y|D = d \sim f(y|d) = \frac{2}{d\pi}\frac{1}{1+(y/d)^2} \cdot \mathbf{1}(-d < y < d)$$
$$D \sim f_D(d) = 2d \cdot \mathbf{1}(0 < d < 1).$$

(a) Give $\text{Corr}(Y, D)$.

We have $\text{Cov}(Y, D) = \mathbb{E}YD - \mathbb{E}Y\mathbb{E}D$. Note that for each $d \in (0, 1)$, the pdf of $Y|D = d$ is symmetric around 0, so $\mathbb{E}[Y|D] = 0$. Therefore $\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|D]) = \mathbb{E}(0) = 0$. Now

$$\mathbb{E}YD = \mathbb{E}(\mathbb{E}[YD|D]) = \mathbb{E}(D\mathbb{E}[Y|D]) = \mathbb{E}(D \cdot 0) = 0,$$

so $\text{Cov}(Y, D) = 0$ and $\text{Corr}(Y, D) = 0$.

(b) Write down an integral expression which will yield the marginal pdf of Y. Give the support of Y.

The support of Y is $(-1, 1)$. For each $y \in (-1, 1)$ the marginal pdf of Y is given by

$$f_Y(y) = \int_{|y|}^{1} \frac{4}{\pi}\frac{1}{1+(y/d)}dd.$$

The integral is a little tricky. If you evaluate it you get

$$\begin{aligned}
f_Y(y) &= \int_{|y|}^{1} \frac{4}{\pi}\frac{1}{1+(y/d)}dd \\
&= \frac{4}{\pi}[d - y\tan^{-1}(d/y)]\Big|_{|y|}^{1} \\
&= \frac{4}{\pi}[(1 - y\tan^{-1}(1/y)) - (|y| - y\tan^{-1}(|y|/y))]\Big|_{|y|}^{1} \\
&= \begin{cases} \frac{4}{\pi}[1 - y(1-\pi/4) - y\tan^{-1}(1/y)], & y > 0 \\ \frac{4}{\pi}[1 + y(1-\pi/4) - y\tan^{-1}(1/y)], & y < 0 \\ \frac{4}{\pi}, & y = 0. \end{cases}
\end{aligned}$$

(c) Let (U, V) be the random variables $U = Y/D$ and $V = D$. Find the joint pdf of (U, V).

Firstly, the joint support of (U, V) is $(-1, 1) \times (0, 1)$. Now we have

$$\begin{matrix} u = y/d =: g_1(y, d) \\ v = d =: g_2(y, d) \end{matrix} \iff \begin{matrix} y = uv =: g_1^{-1}(u, v) \\ d = v =: g_2^{-1}(u, v) \end{matrix}$$

with Jacobian

$$J(u, v,) = \begin{vmatrix} \frac{d}{du}uv & \frac{d}{dy}uv \\ \frac{d}{du}v & \frac{d}{dv}v \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

So the joint pdf of (U, V) is given by

$$f(u, v) = \frac{4}{\pi} \frac{1}{1 + u^2} \cdot v \cdot \mathbf{1}(-1 < u < 1, 0 < v < 1).$$

(d) Give $\text{Cov}(U, V)$.

Since the joint pdf of (U, V) can be factored into the product of a function of u and a function of v, the random variables U and V are independent. Therefore $\text{Cov}(U, V) = 0$.